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Functional and generalized separable solutions to unsteady Navier–Stokes equations

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ABSTRACT

The paper studies unsteady Navier–Stokes equations with two space variables. It shows that the non-linear fourth-order equation for the stream function with three independent variables admits functional separable solutions described by a system of three partial differential equations with two independent variables. The system is found to have a number of exact solutions, which generate new classes of exact solutions to the Navier–Stokes equations. All these solutions involve two or more arbitrary functions of a single argument as well as a few free parameters. Many of the solutions are expressed in terms of elementary functions, provided that the arbitrary functions are also elementary; such solutions, having relatively simple form and presenting significant arbitrariness, can be especially useful for solving certain model problems and testing numerical and approximate analytical hydrodynamic methods. The paper uses the obtained results to describe some model unsteady flows of viscous incompressible fluids, including flows through a strip with permeable walls, flows through a strip with extrusion at the boundaries, flows onto a shrinking plane, and others. Some blow-up modes, which correspond to singular solutions, are discussed.

1. Introduction. The stream function equation

1.1. Preliminary remarks

The Navier–Stokes equations and other hydrodynamic equations are important and fairly common in various areas of science and engineering (e.g., see [1–4]).

Exact solutions to the Navier–Stokes and related equations contribute to better understanding of qualitative features of steady and unsteady fluid flows; these features include stability, non-uniqueness, spatial localization, blow-up, and others. Exact solutions to the Navier–Stokes equations allow efficient estimates of the domain of applicability for simplified hydrodynamic models, including boundary-layer equations and Euler equations.

Exact solutions with significant functional arbitrariness are of particular interest because they may be used as test problems for assessing the accuracy of numeric, asymptotic, and approximate analytical methods for solving suitable non-linear hydrodynamic-type PDEs as well as certain model problems.

1.2. The concepts of 'exact solution' and 'linearizing solution' for non-linear PDEs

In what follows, the term 'exact solution' with regard to non-linear partial differential equations (including the Navier–Stokes equations) is used in the following cases [5,6]:

(i) the solution is expressible in terms of elementary functions or in closed form with definite or/and indefinite integrals;
(ii) the solution is expressible in terms of solutions to ordinary differential equations (or systems of such equations);
(iii) combinations of the first two items are also allowed.

Apart from exact solutions, we will also be dealing with 'linearizing solutions', which are expressible in terms of solutions to linear partial differential equations, perhaps in conjunction with solutions (i) and (ii).

Remark 1. To find exact solutions to the Navier–Stokes, boundary-layer, and related equations, one usually employs the classical method for symmetry reductions [7–13] (based on the Lie group

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analysis of PDEs), direct method for symmetry reductions [14–19] (also known as the Clarkson–Kruskal direct method), non-classical method for symmetry reductions [20–22], and method of generalized separation of variables [5,23–29]. For some other, less common methods, see also [30–36]. Extensive surveys of exact solutions to the Navier–Stokes and boundary-layer equations can be found in [5,4,37,38].

1.3. Navier–Stokes equations with two space variables. Reduction to the stream function equation

The unsteady Navier–Stokes equations with two space variables are written as

$$U_t+UU_x+VU_y=-P_x+\nu\Delta U,\\ V_t+UV_x+VV_y=-P_y+\nu\Delta V,\\ U_x+V_y=0, \tag{1}$$

where t is time, x and y are Cartesian coordinates, U and V are the fluid velocity components, P is the fluid pressure-to-density ratio, ν is the kinematic viscosity, and Δ is the Laplace operator.

By introducing a stream function w defined by the formulas

$$U=w_y,\quad V=-w_x \tag{2}$$

followed by eliminating the normalized pressure P, one can reduce system (1) to a single non-linear fourth-order equation [1,2,4]:

$$(\Delta w)_t+w_y(\Delta w)_x-w_x(\Delta w)_y=\nu\Delta\Delta w,\quad \Delta w=w_{xx}+w_{yy}. \tag{3}$$

For steady and unsteady exact solutions to the two- and three-dimensional Navier–Stokes equations, see the studies [1,2,4,5,7,10,13,20,21,26,27,30,31,34,35,37–54] and references therein. Some previous results related to the present paper will be discussed below in Remark 3). For models and exact solutions to hyperbolic and differential-difference Navier–Stokes equations, see [28,55–57]. A number of steady and unsteady exact solutions to boundary layer equations, which are related asymptotic equations derived from the Navier–Stokes equations at large Reynolds numbers, can be found, for example, in [1,2,5,8,9,12,14–19,22–25,32,33,58–61].

1.4. Generalized and functional separable solutions

The study [26] dealt with the stream function equation (3) and presented a number of its generalized separable solutions of the form

$$w=\sum_{k=1}^{n} f_k(x)g_k(y,t)\quad \text{or}\quad w=\sum_{k=1}^{n} f_k(x,t)g_k(y). \tag{4}$$

The functions $f_k(x)$ and $g_k(y,t)$ (or $f_k(x,t)$ and $g_k(y)$) are determined in the analysis of the equation resulting from inserting (4) into (3).

The first solution in (4) most frequently involves the following functions:

$$f_k(x)=x^m\quad (m=0,1,2),\quad f_k(x)=\exp(\lambda_k x),\\ f_k(x)=\cos(\beta_k x),\quad f_k(x)=\sin(\beta_k x),$$

where λ_i and β_i are unknown parameters. The other set of functions, $g_k(y,t)$, is determined by solving the corresponding non-linear equations.

The books [5,6,62] detail various modifications of the method of generalized separation of variables based on seeking solutions of the form (4). These books give a large number of non-linear PDEs and systems of PDEs, including the Navier–Stokes equations and the stream function equation (3), that admit generalized separation of variables.

This paper will present more-complex functional separable solutions dependent, in a special way, on the original independent variables as well as the extra variable $z=\varphi(t)x+\psi(t)y$; see the subsequent section for details.

Functional separable solutions to non-linear hydrodynamic-type and diffusion-type equations can be found, for example, in [18,60,63–71]. The books [5,6] describe various modifications of the method of functional separation of variables and give specific examples of its usage.

Remark 2. For higher-order non-linear hydrodynamic PDEs such as Eq. (3), the direct methods of generalized and functional separation of variables (with a preset form of exact solutions involving arbitrary functions) usually suggest easier calculations and result in simpler equations than the non-classical method of symmetry reductions based on invariant surface conditions and the method of differential constraints. Moreover, the fact that generalized and functional solutions (as well as those obtained using Clarkson–Kruskal direct reductions [72]) can be represented in terms of differential constraints [73] has no practical value (see Section 34.5 in [5]).

It is noteworthy that there is a recent modification of the method of functional separation of variables [60,61] which has proved to be effective for constructing exact solutions to unsteady axisymmetric boundary-layer equations.

1.5. Boundary conditions in some hydrodynamic problems

In subsequent sections, we will give examples of using the obtained exact solutions to construct solutions of several unsteady model hydrodynamic problems.

In Section 4, we will look at hydrodynamic problems with different types of boundary conditions for the velocity components (e.g., see [2,4,38,41,50,52]).

Surface stretching or shrinking (extrusion) are described by the conditions

$$U=\xi(t,x),\quad V=0\quad \text{at } y=0.$$

The trivial case of $\xi(t,x)=0$ corresponds to the no-slip conditions at a fixed surface. Functions $\xi(t,x)=A(t)$ correspond to a rigid surface moving in its own plane (along the x-axis). Unsteady stretching or shrinking of a surface is usually modeled by a linear function in the space coordinate, $\xi(t,x)=A(t)x$, with $A>0$ corresponding to stretching and $A<0$ corresponding to shrinking.

Feeding or removing a fluid through a permeable (porous) surface is characterized by the conditions

$$U=0,\quad V=\eta(t,x)\quad \text{at } y=0.$$

Uniform feeding or removal of a fluid through a permeable wall is modeled by $\eta(t,x)=B(t)$, with $B>0$ corresponding to feeding and $B<0$, to removal. The case $\eta(t,x)=\text{const}$ corresponds to steady-state feeding/removal.

2. Form of functional separable solutions. The determining system of equations

2.1. General form of desired functional separable solutions

We look for exact solutions to Eq. (3) of the form

$$w=xf(t,z)+yg(t,z)+h(t,z)+\tfrac{1}{2}a(t)x^2+b(t)xy+\tfrac{1}{2}c(t)y^2,\\ z=\varphi(t)x+\psi(t)y, \tag{5}$$

where $f=f(t,z)$, $g=g(t,z)$, $h=h(t,z)$, $a=a(t)$, $b=b(t)$, $c=c(t)$, $\varphi=\varphi(t)$, and $\psi=\psi(t)$ are unknown functions to be determined in the analysis. In the special case $\varphi=0$ or $\psi=0$, formula (5) defines generalized separable solutions.

Remark 3. The representation (5) is a generalization for a number of exact solutions obtained previously. In particular, solutions of the form (5) with

$$f(t,z)=f_0(t),\quad g(t,z)=g_0(t),\quad a=b=c=0,$$
$$\varphi(t)=k=\text{const},\quad \psi(t)=\lambda=\text{const}$$

were treated in [4,26,45,46,51].

Solutions of the form (5) with

$$g(t,z)=h(t,z)=0,\quad a=b=c=0,\quad \varphi(t)=0,\quad \psi(t)=1$$

describe different flow modes near a stagnation point as well as near a stretching or shrinking plane (extrusion processes). Such solutions were studied, for example, in [4,26,38–41,44,47,48,50,52,54].

Solutions with

$$g(t,z)=0,\quad a=b=c=0,\quad \varphi(t)=k=\text{const},\quad \psi(t)=1,$$

including $k=0$ and $k\neq 0$, were treated in [5,26].

A solution of the form (5) with

$$f(t,z)=-\psi(t)g(t,z),\quad a(t)=\frac{\psi'_t}{2(1+\psi^2)},$$
$$b(t)=2a(t)\psi,\quad c(t)=-a(t),\quad \varphi(t)=1$$

was described in [21].

A most comprehensive survey of known exact solutions to Eq. (3) defined by special cases of formula (5) can be found in [5].

2.2. The determining system of equations

Substituting (5) into the stream function equation (3) gives

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \tag{6}$$

where A, B, C, D, E, and F are functional coefficients dependent on only t and z; see the appendix for the expressions of these coefficients. For Eq. (6) to be satisfied for any x and y, all functional coefficients must be set equal to zero:

$$A=B=C=D=E=F=0. \tag{7}$$

The first three equations, $A=B=C=0$, are satisfied if

$$\varphi'_t+b\varphi-a\psi=0,\quad \psi'_t+c\varphi-b\psi=0. \tag{8}$$

Assuming here and henceforth the three functions of time $b=b(t)$, $\varphi=\varphi(t)$, and $\psi=\psi(t)$ to be arbitrary, we find $a=a(t)$ and $c=c(t)$ as

$$a=\frac{\varphi'_t+b\varphi}{\psi},\quad c=\frac{b\psi-\psi'_t}{\varphi}. \tag{9}$$

These formulas are valid if $\varphi\not\equiv 0$ and $\psi\not\equiv 0$. If $\varphi\equiv 0$ or $\psi\equiv 0$, we have

$$\varphi\equiv 0,\quad a=0,\quad b=\psi'_t/\psi,\quad c=c(t)\ \text{is an arbitrary function;}$$
$$\psi\equiv 0,\quad c=0,\quad b=-\varphi'_t/\varphi,\quad a=a(t)\ \text{is an arbitrary function.}$$

Using relations (8) and making some rearrangements, one can reduce the last three equations in (7), $D=E=F=0$, to the following non-linear system of partial differential equations for f, g, and h:

$$f_{tzz}+(s+b)f_{zz}-ag_{zz}+\psi(f_zf_{zz}-ff_{zzz})+\varphi(gf_{zzz}-f_zg_{zz})=\nu(\varphi^2+\psi^2)f_{zzzz}, \tag{10}$$

$$g_{tzz}+(s-b)g_{zz}+cf_{zz}+\varphi(gg_{zzz}-g_zg_{zz})+\psi(g_zf_{zz}-fg_{zzz})=\nu(\varphi^2+\psi^2)g_{zzzz}, \tag{11}$$

$$\begin{aligned}&\overline{h}_{tzz}+(\psi f_{zz}-\varphi g_{zz})\overline{h}_z+(\varphi g-\psi f)\overline{h}_{zzz}+a'_t+c'_t+2\varphi'_tf_z+2\psi'_tg_z+2\varphi f_{tz}\\&\quad+2\psi g_{tz}+[(3\varphi^2+\psi^2)g-2\varphi\psi f]f_{zz}+[2\varphi\psi g-(\varphi^2+3\psi^2)f]g_{zz}\\&\quad-4\nu(\varphi^2+\psi^2)(\varphi f_{zzz}+\psi g_{zzz})=\nu(\varphi^2+\psi^2)\overline{h}_{zzzz},\end{aligned} \tag{12}$$

where

$$s=s(t)=\frac{(\varphi^2+\psi^2)'_t}{\varphi^2+\psi^2},\quad \overline{h}=(\varphi^2+\psi^2)h. \tag{13}$$

The system of equations (10)–(12) will be referred to as the *determining system*. It splits into a subsystem of two coupled equations (10) and (11) for f and g as well as a passive equation (12), which is linear in h and does not affect f or g. System (10)–(12) involves three arbitrary functions of time, $b=b(t)$, $\varphi=\varphi(t)$, and $\psi=\psi(t)$, with the coefficients a, c, and s expressed in terms of these functions according to (9) and (13).

In what follows, we will be looking for exact solutions to the determining system (10)–(12), which generate exact solutions of the form (5) to the stream function equation (3). The functional coefficients a and c in system (10)–(12) are defined by (9).

Along with Eq. (12), we will also be using the equivalent equation

$$\begin{aligned}&\overline{h}_{tzz}+\psi(f_{zz}\overline{h}_z-f\overline{h}_{zzz})+\varphi(g\overline{h}_{zzz}-g_{zz}\overline{h}_z)+a'_t+c'_t+2(\varphi f+\psi g)_{tz}\\&\quad+2(\varphi g-\psi f)(\varphi f_{zz}+\psi g_{zz})+(\varphi^2+\psi^2)(gf_{zz}-fg_{zz})\\&\quad-4\nu(\varphi^2+\psi^2)(\varphi f_{zzz}+\psi g_{zzz})=\nu(\varphi^2+\psi^2)\overline{h}_{zzzz},\end{aligned} \tag{14}$$

which is often more convenient in calculations.

We would like to emphasize that exact solutions to subsystem (10), (11) generate linearizing solutions to system (10)–(12) and, consequently, the stream function equation (3).

2.3. Formulas allowing generalizations of exact solutions to subsystem (10), (11)

The subsystem of the non-linear coupled equations (10) and (11) has a remarkable property which is stated below as a theorem.

Theorem 1. *Suppose that the functions $f(t,z)$ and $g(t,z)$ solve the coupled system (10), (11). Then the functions*

$$f_1=f(t,z+\xi)+\eta,\quad g_1=g(t,z+\xi)+\zeta,\quad \xi=\int(\psi\eta-\varphi\zeta)\,dt, \tag{15}$$

with arbitrary $\eta=\eta(t)$ and $\zeta=\zeta(t)$, also solve this system.

This theorem can be proved by direct verification.

Theorem 1 makes it possible to generalize exact solutions of the non-linear coupled equations (10) and (11) by including additional arbitrary functions.

3. Solutions to the determining system corresponding to degenerate solutions of equations (10) and (11)

3.1. Solutions with f and g linear in z

Eqs. (10) and (11) can be satisfied identically with degenerate solutions of the form

$$f=f_1(t)z+f_0(t),\quad g=g_1(t)z+g_0(t), \tag{16}$$

where $f_1=f_1(t)$, $f_0=f_0(t)$, $g_1=g_1(t)$, and $g_0=g_0(t)$ are arbitrary functions. Substituting (16) into (12) yields the equation

$$\begin{aligned}&\overline{h}_{tzz}+[(\varphi g_1-\psi f_1)z+\varphi g_0-\psi f_0]\overline{h}_{zzz}+(a+c+2\varphi f_1+2\psi g_1)'_t\\&=\nu(\varphi^2+\psi^2)^2\overline{h}_{zzzz},\end{aligned} \tag{17}$$

which involves seven arbitrary functions of time: φ, ψ, f_0, f_1, g_0, g_1, and b (recall that a and c are expressed in terms of φ, ψ, and b by formulas (9)). The substitution

$$\xi=\overline{h}_{zz}+a+c+2\varphi f_1+2\psi g_1 \tag{18}$$

reduces Eq. (17) to a linear parabolic second-order equation with variable coefficients:

$$\xi_t+[(\varphi g_1-\psi f_1)z+\varphi g_0-\psi f_0]\xi_z=\nu(\varphi^2+\psi^2)\xi_{zz}. \tag{19}$$

This sort of equation was treated in [74]. The transformation

$$\xi=\zeta(\tau,\eta),\quad \tau=\nu\int\sigma^2(\varphi^2+\psi^2)\,dt,\quad \eta=\sigma z+\int\sigma(\psi f_0-\varphi g_0)\,dt,$$

$$\sigma=\sigma(t)=\exp\left[\int(\psi f_1-\varphi g_1)\,dt\right], \tag{20}$$

reduces Eq. (19) to the classical heat equation

$$\zeta_\tau=\zeta_{\eta\eta}. \tag{21}$$

3.2. Solutions with f and g quadratic in z

Eqs. (10) and (11) admit degenerate solutions of the form

$$f=f_2(t)z^2+f_1(t)z+f_0(t),\quad g=g_2(t)z^2+g_1(t)z+g_0(t), \tag{22}$$

where $f_1=f_1(t)$, $f_0=f_0(t)$, $g_1=g_1(t)$, and $g_0=g_0(t)$ are arbitrary functions,

$$f_2(t)=\frac{K\varphi}{\varphi^2+\psi^2},\quad g_2(t)=\frac{K\psi}{\varphi^2+\psi^2}, \tag{23}$$

and K is an arbitrary constant. Eq. (12) determined by the functions of (22) and (23) can be reduced to the classical heat equation [74].

It can be shown that solution (5) with f and g defined by (22) is equivalent, up to redefining $h=h(t,z)$, to solution (5) with f and g defined by (16).

Remark 4. Searching for solutions of the form (5) with f and g cubic in z eventually leads to solution (5) with f and g defined by (16) and a modified $h=h(t,z)$.

4. Solutions to the determining system of special form consisting of two equations. Solutions to some hydrodynamic problems

4.1. A special case where the determining system reduces to two equations

Consider the special case of Eq. (5) with

$$g=0,\quad a=b=c=0,\quad \varphi=0,\quad \psi=1,$$

which corresponds to the stream function

$$w=xf(t,y)+g(t,y). \tag{24}$$

In this case, system (10)–(12) simplifies significantly to become a system of two rather than three equations:

$$f_{tyy}+f_yf_{yy}-ff_{yyy}=\nu f_{yyyy}, \tag{25}$$

$$h_{tyy}+h_yf_{yy}-fh_{yyy}=\nu h_{yyyy}. \tag{26}$$

A most comprehensive survey of exact solutions to the time-dependent equation (25) and system (25), (26) can be found in [5] (see also [4,26,37,38]). Below we present generalizations of some known solutions and consider a few new problems.

Eq. (25) contains one unknown function, f, and is independent of Eq. (26).

4.2. Two theorems on exact solutions to the determining system

The following theorem holds true.

Theorem 2. *Let $f=f(t,y)$ be a solution to Eq. (25). Then Eq. (26) admits the exact solution*

$$h=Cf_y+A(t)f-A'_t(t)y, \tag{27}$$

where C is an arbitrary constant and $A=A(t)$ is an arbitrary function.

This theorem can be verified by substituting expression (27) into Eq. (26) and taking into account Eq. (25) and the equation obtained by differentiating (25) with respect to y:

$$f_{tyyy}+f_{yy}^2-ff_{yyyy}=\nu f_{yyyyy}.$$

Formula (27) allows one to construct exact solutions to Eq. (26) whenever a solution to Eq. (25) is known.

The following, more general statement also holds true.

Theorem 3. *Let $f=f(t,y)$ be a solution to Eq. (25). Then system (25), (26) admits the exact solution*

$$f_1=f(t,y+B)+B'_t,$$
$$h_1=Cf_y(t,y+B)+Af(t,y+B)-A'_ty, \tag{28}$$

where C is an arbitrary constant, while $A=A(t)$ and $B=B(t)$ are arbitrary functions.

In particular, given a time-invariant solution to Eq. (25), Theorem 3 allows one to construct time-dependent solutions to system (25), (26) involving two arbitrary functions of time and an arbitrary constant.

Moreover, the above remains valid if one adds an arbitrary function of time, which does not affect the velocity components (2), to the right-hand side of formula (27) and the second formula in (28).

4.3. A solution to system (25), (26) rational in y

One can verify by direct substitution that Eq. (25) admits the time-invariant solution $f=6\nu/y$. By the formulas (27), we get the following exact solution to system (25), (26):

$$f=\frac{6\nu}{y+B}+B'_t,\quad h=\frac{C_1}{(y+B)^2}+\frac{6\nu A}{y+B}-A'_ty.$$

It involves two arbitrary functions, $A=A(t)$ and $B=B(t)$, and an arbitrary constant, $C_1=-6\nu C$.

4.4. Solutions of system (25), (26) involving an exponential of y. Examples of solving some problems

System (25), (26) admits exact solutions of the form

$$f=a(t)e^{-\lambda(t)y}+b(t)y+c(t),$$
$$h=\alpha(t)e^{-\lambda(t)y}+\beta(t)y, \tag{29}$$

with the six functional coefficients $a=a(t)$, $b=b(t)$, $c=c(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\lambda=\lambda(t)$ satisfying the following three equations:

$$\lambda'_t-b\lambda=0,$$
$$a'_t+3ab+ac\lambda-\nu a\lambda^2=0,$$
$$\alpha'_t+2b\alpha+a\beta+c\alpha\lambda-\nu\alpha\lambda^2=0. \tag{30}$$

The second and third equations have been rearranged using the first one. The functions a, α, and λ in (30) can be treated as arbitrary. Then the other three functions can be found without integrals:

$$b=\frac{\lambda'_t}{\lambda},\quad c=-\frac{1}{a\lambda}(a'_t+3ab)+\nu\lambda,$$
$$\beta=\frac{1}{a}(-\alpha'_t-2b\alpha-c\alpha\lambda+\nu\alpha\lambda^2). \tag{31}$$

Below we give a few examples illustrating the usage of the above formulas for constructing solutions to some model hydrodynamic problems.

Example 1. Let us look at the special case of solution (29) with

$$\lambda=\text{const},\quad b=0,\quad c=\nu\lambda-\frac{a'_t}{a\lambda},\quad \alpha=a\sigma,\quad \beta=-\sigma'_t,\tag{32}$$

where $a=a(t)$ and $\sigma=\sigma(t)$ are arbitrary functions. The expressions (32) satisfy system (30) and follow from (31).

Substituting (32) into (29) and taking into account (24), we arrive at the stream function

$$w=x\left(ae^{-\lambda y}+\nu\lambda-\frac{a'_t}{a\lambda}\right)+a\sigma e^{-\lambda y}-\sigma'_t y.\tag{33}$$

The velocity components are obtained by formulas (2):

$$U=-a\lambda xe^{-\lambda y}-a\sigma\lambda e^{-\lambda y}-\sigma'_t,\quad V=-ae^{-\lambda y}-\nu\lambda+\frac{a'_t}{a\lambda}.\tag{34}$$

We set $y=0$ to obtain

$$U|_{y=0}=-a\lambda x-a\sigma\lambda-\sigma'_t,\quad V|_{y=0}=-a-\nu\lambda+\frac{a'_t}{a\lambda}.\tag{35}$$

Now we choose the free functions a and σ such that

$$a=-\frac{\nu\lambda}{C_1\exp(\nu\lambda^2t)+1},\quad \sigma=\frac{C_2\exp(\nu\lambda^2t)}{C_1\exp(\nu\lambda^2t)+1},\tag{36}$$

where C_2 is an arbitrary constant and C_1 is an arbitrary constant such that $C_1\ge 0$ or $C_1<-1$. Then the boundary relations (35) significantly simplify to become

$$U|_{y=0}=A(t)x,\quad V|_{y=0}=0,\tag{37}$$

where

$$A(t)=\frac{\nu\lambda^2}{C_1\exp(\nu\lambda^2t)+1}.$$

For $\lambda>0$, formulas (34) in conjunction with (36) describe three-parameter unsteady modes of flow in the half-plane $0\le y<\infty$ caused by stretching (if $A>0$) or shrinking (if $A<0$) of the surface $y=0$ according to the law (37) under special initial conditions (corresponding to $t=0$ in Eq. (36)). In the special case $C_1=C_2=0$, the above formulas lead to the steady-state solution of [41], which models extrusion. If $C_2=0$ and $C_1\ne 0$, formulas (34) and (36) define solution [38].

Example 2. Setting $\alpha=\beta=0$ and substituting (29) and (31) into (24), we obtain the stream function

$$w=x\left(ae^{-\lambda y}+\frac{\lambda'_t}{\lambda}y+\nu\lambda-3\frac{\lambda'_t}{\lambda^2}-\frac{a'_t}{a\lambda}\right).\tag{38}$$

The corresponding velocity components are expressed as

$$U=x\left(-a\lambda e^{-\lambda y}+\frac{\lambda'_t}{\lambda}\right),$$
$$V=-ae^{-\lambda y}-\frac{\lambda'_t}{\lambda}y-\nu\lambda+3\frac{\lambda'_t}{\lambda^2}+\frac{a'_t}{a\lambda}\tag{39}$$

with arbitrary $a=a(t)$ and $\lambda=\lambda(t)$. Solution (39) satisfies the following boundary conditions as $y\to\infty$:

$$U\to\Lambda(t)x,\quad V\to-\Lambda(t)y,\quad \text{where } \Lambda=\lambda'_t/\lambda.$$

These conditions are used to model viscous flows about a stagnant point (e.g., see [4,38,50,52,75]).

1°. *Fluid flow onto a shrinking plane*: Let us look at the special case

$$a=\frac{a_0}{\sqrt{t+C}},\quad \lambda=\frac{\lambda_0}{\sqrt{t+C}},\quad a_0=-\frac{1}{\lambda_0}(\nu\lambda_0+2),\tag{40}$$

where $C>0$ and $\lambda_0>0$ are arbitrary constants. At the surface $y=0$, solution (39), (40) satisfies the conditions

$$U|_{y=0}=-A(t)x,\quad V|_{y=0}=0,\tag{41}$$

where

$$A(t)=-\frac{2\nu\lambda_0^2+3}{t+C}.$$

Hence, solution (39), (40) describes an unsteady flow onto a shrinking plane.

2°. *Fluid flow onto a fixed solid surface* (*blow-up*): Let us require that solution (39) satisfy the no-slip condition at the boundary $y=0$: $U|_{y=0}=V|_{y=0}$. This results in a system of ODEs for $a=a(t)$ and $\lambda=\lambda(t)$:

$$-a\lambda+\frac{\lambda'_t}{\lambda}=0,\quad -a-\nu\lambda+3\frac{\lambda'_t}{\lambda^2}+\frac{a'_t}{a\lambda}=0.$$

Eliminating $a=\lambda^{-2}\lambda'_t$ yields a second-order non-linear ODE for λ:

$$\lambda''_{tt}=\nu\lambda^2\lambda'_t.\tag{42}$$

A one-parameter particular solution to Eq. (42) is expressed as

$$\lambda=k(C_1-t)^{-1/2},\quad k=(2\nu/3)^{-1/2},\tag{43}$$

where $C_1>0$ is an arbitrary constant. The function (43) is a real function defined on the bounded time interval $0\le t<C_1$ and associated with a blow-up [76,77], since $\lambda\to\infty$ as $t\to C_1$.

Then the corresponding velocity components (39) become

$$U=\frac{\lambda'_t}{\lambda}x(1-e^{-\lambda y})=\frac{\lambda^2}{2k^2}x(1-e^{-\lambda y}),$$
$$V=\frac{\lambda'_t}{\lambda^2}(1-e^{-\lambda y})-\frac{\lambda'_t}{\lambda}y=\frac{\lambda}{2k^2}x(1-e^{-\lambda y})-\frac{\lambda^2}{2k^2}y.\tag{44}$$

Solution (44) does not have singularities at the initial time $t=0$ and is infinitely differentiable with respect to the spatial coordinates x and y. For $0\le t<C_1$ and far away from a fixed surface, as $y\to\infty$, formulas (44) describe a linear shear flow.

3°. *Flow onto a fixed rigid surface* (*blow-up*): A first integral of Eq. (42) is $\lambda'_t=\frac{1}{3}\nu(\lambda^3+C_2^3)$, which is a separable equation. Integrating it gives the following general solution in implicit form:

$$t=\frac{1}{2\nu C_2^2}\ln\frac{(\lambda+C_2)^2}{\lambda^2-C_2\lambda+C_2^2}+\frac{\sqrt{3}}{\nu C_2^2}\arctan\frac{2\lambda-C_2}{\sqrt{3}\,C_2}+C_1,\tag{45}$$

where C_1 and C_2 are arbitrary constants. For the exponential terms in solution (39) to die away, one has to assume that $\lambda>0$. The argument of the logarithmic function in Eq. (42) attaints its maximum equal to 4 at $\lambda=C_2$, while arctangent is a bounded function that does not exceed $\pi/2$. Therefore, with fixed C_1 and $C_2\ne 0$ (the case $C_2=0$ was discussed above in Item 1°), time t can only assume bounded positive values, which corresponds to a blow-up.

4.5. Solutions of system (25), (26) involving trigonometric functions of y. Examples of solving some problems

System (25), (26) admits exact solutions of the form

$$f=a(t)\cos[\lambda(t)y+\sigma(t)]+b(t)y+c(t),$$
$$h=\alpha(t)\cos[\lambda(t)y+\sigma(t)]+s(t)\sin[\lambda(t)y+\sigma(t)]+\beta(t)y,\tag{46}$$

with the functional coefficients $a=a(t)$, $b=b(t)$, $c=c(t)$, $s=s(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, $\lambda=\lambda(t)$, and $\sigma=\sigma(t)$ satisfying the following five equations:

$$\lambda'_t-b\lambda=0,$$
$$\sigma'_t-c\lambda=0,$$
$$a'_t+3ab+\nu a\lambda^2=0,$$
$$\alpha'_t+2b\alpha+a\beta+\nu\alpha\lambda^2=0,$$
$$s'_t+2bs+\nu s\lambda^2=0.\tag{47}$$

The last three equations have been rearranged using the first two equations.

The functions λ, σ, and α can be treated as arbitrary. Then the other five functions are determined as follows:

$$a=\frac{a_0}{\lambda^3}\exp\left(-\nu\int\lambda^2\,dt\right),\quad b=\frac{\lambda_t'}{\lambda},\quad c=\frac{\sigma_t'}{\lambda},$$
$$\beta=-\frac{1}{a}(\alpha_t'+2b\alpha+\nu\alpha\lambda^2),\quad s=\frac{s_0}{\lambda^2}\exp\left(-\nu\int\lambda^2\,dt\right),\tag{48}$$

where a_0 and s_0 are arbitrary constants.

In what follows, we give a few examples of how the above formulas can be used to construct solutions to some model hydrodynamic problems.

Example 3. Let us look at the special case of solution (46) with

$$\lambda=\text{const},\quad \sigma=\text{const},\quad b=c=0,\quad a=a_1E(t),\quad \alpha=a_1E(t)\omega(t),$$
$$s=s_1E(t),\quad E(t)=\exp(-\nu\lambda^2t),\quad \beta=-\omega_t',\tag{49}$$

where $a_1=a_0\lambda^{-3}$ and $s_1=s_0\lambda^{-2}$ are arbitrary constants, while $\omega=\omega(t)$ is an arbitrary function. The expressions (49) satisfy system (47) and follow from formulas (48).

Substituting (49) into (46) and taking into account relations (5) and (24), we arrive at the stream function

$$w=a_1E(t)\cos(\lambda y+\sigma)[x+\omega(t)]+s_1E(t)\sin(\lambda y+\sigma)-\omega_t'(t)y,\tag{50}$$

where a_1, s_1, λ, and σ are arbitrary constants and $\omega=\omega(t)$ is an arbitrary function. From formulas (2), we find the velocity components

$$U=-a_1\lambda E(t)\sin(\lambda y+\sigma)[x+\omega(t)]+s_1\lambda E(t)\cos(\lambda y+\sigma)-\omega_t'(t),$$
$$V=-a_1E(t)\cos(\lambda y+\sigma),\quad E(t)=\exp(-\nu\lambda^2t).\tag{51}$$

Consider the following two cases.

1°. *Case $\sigma=0$ (flow in a strip with permeable boundaries)*: By setting $y=0$ in (51) and introducing a new function $\theta=\theta(t)$ instead of $\omega=\omega(t)$ such that

$$\omega(t)=-\int\theta(t)\,dt-\frac{s_1}{\nu\lambda}E(t)+C_1,$$

we obtain

$$U|_{y=0}=\theta(t),\quad V|_{y=0}=-a_1E(t).\tag{52}$$

The first condition in (52) suggests that the boundary $y=0$ moves as a rigid body along the x-axis according to the arbitrary law $\theta(t)$ (in particular, it performs periodic oscillations if θ is periodic); the trivial case $\theta\equiv0$ corresponds to a stationary boundary. The second condition in (52) suggests that fluid is supplied or withdrawn through the (permeable) surface, depending on the sign of a_1, at a rate exponentially decreasing with time. Solution (51) is periodic in y. This implies that the formulas (51) describe a fluid flow in the strip $0\le y\le 2\pi/\lambda$, with identical conditions of the form (52) set at the boundaries for the velocity components.

2°. *Case $\sigma=\pi/2$ (flow in a strip with boundary extrusion)*: At $\sigma=\pi/2$ and $\omega=0$, formulas (51) become

$$U=-a_1\lambda E(t)\cos(\lambda y)x-s_1\lambda E(t)\sin(\lambda y),$$
$$V=a_1E(t)\sin(\lambda y),\quad E(t)=\exp(-\nu\lambda^2t).\tag{53}$$

By setting $y=0$, we get

$$U|_{y=0}=-a_1\lambda E(t)x,\quad V|_{y=0}=0.$$

These conditions suggest that the boundary $y=0$ stretches if $a_1\lambda<0$ or shrinks if $a_1\lambda>0$. Solution (53) is periodic in y. It follows that the formulas (53) describe a flow in the strip $0\le y\le 2\pi/\lambda$ whose boundaries are deformed in a concerted fashion (e.g., during extrusion).

Example 4. By setting $s_1=0$ in (53) and renaming $x\rightleftarrows y$ and $U\rightleftarrows V$, we obtain

$$U=a_1E(t)\sin(\lambda x),\quad V=-a_1\lambda E(t)y\cos(\lambda x).\tag{54}$$

Further, by setting $y=0$, we get

$$U|_{y=0}=a_1E(t)\sin(\lambda x),\quad V|_{y=0}=0.\tag{55}$$

These relations suggest that the boundary $y=0$ deforms, stretches or shrinks, periodically in x, with the deformation amplitude decaying exponentially with time.

Example 5. With $\alpha=s=\beta=0$ and $\sigma=\pi/2$ and in view of (30), the stream function defined by (24) becomes

$$w=x\left[-a\sin(\lambda y)+\frac{\lambda_t'}{\lambda}y\right],\tag{56}$$

where $\lambda=\lambda(t)$ is an arbitrary function. The corresponding fluid velocity components are

$$U=x\left[-a\lambda\cos(\lambda y)+\frac{\lambda_t'}{\lambda}\right],\quad V=a\sin(\lambda y)-\frac{\lambda_t'}{\lambda}y.\tag{57}$$

The function $a=a(t)$ is related to λ through the third equation in (47), which can be rewritten as

$$\frac{a_t'}{a}+3\frac{\lambda_t'}{\lambda}+\nu\lambda^2=0.\tag{58}$$

1°. *Flow in the first quadrant dependent on free parameters*: We require the no-slip condition at the surface $y=0$. This results in the ODE

$$\lambda_t'=a\lambda^2.\tag{59}$$

Eliminating a from Eqs. (58) and (59) yields the following second-order non-linear ODE for $\lambda=\lambda(t)$:

$$\lambda\lambda_{tt}''+(\lambda_t')^2+\nu\lambda^3\lambda_t'=0.\tag{60}$$

A first integral of this equation is $\lambda\lambda_t'+\frac14\nu\lambda^4=C_1$. Integrating further gives the solution

$$\lambda=\pm\sqrt{k}\left(\frac{C_2e^{\nu kt}-1}{C_2e^{\nu kt}+1}\right)^{1/2},\tag{61}$$

where C_2 and $k>0$ are arbitrary constants ($C_1=\frac12\nu k^2>0$). Formulas (61) make sense for any $t\ge0$ as long as $|C_2|>1$. By letting $t\to\infty$, we get $\lambda\to\pm\sqrt{k}$. In view of (59), the velocity components (57) can be written as

$$U=\frac{\lambda_t'}{\lambda}x[1-\cos(\lambda y)],\quad V=\frac{\lambda_t'}{\lambda^2}[\sin(\lambda y)-\lambda y].\tag{62}$$

At the surface $x=0$, we have $U|_{x=0}=0$ and $V|_{x=0}=V_0(t,x)$, with $V_0(t,x)$ determined by the right-hand side of the second relation in (62). It follows that the formulas (61) and (62) describe a flow in the first quadrant ($x\ge0$, $y>0$) due to special stretching/shrinking of the boundary $x=0$ and with fixed boundary $y=0$. The right-hand side of the formula for V in (62) can be treated as a superposition of linear and oscillatory extrusion.

2°. *Flow in the first quadrant dependent on an arbitrary function*: Since $a=a(t)$ and $\lambda=\lambda(t)$ are connected by a single differential constraint (58), suggesting that either can be considered arbitrary, these functions can be selected so that the velocity components (57) at the stretching/shrinking surface satisfy the conditions $U|_{y=0}=\omega(t)x$ and $V|_{y=0}=0$, where $\omega=\omega(t)$ is an arbitrary function. In the special case $\omega=\text{const}$, we get an unsteady solution satisfying steady-state conditions of extrusion.

4.6. Exact solutions involving hyperbolic functions of y

1°. System (25), (26) admits exact solutions of the form

$$f=a(t)\cosh[\lambda(t)y+\sigma(t)]+b(t)y+c(t),$$
$$h=\alpha(t)\cosh[\lambda(t)y+\sigma(t)]+s(t)\sinh[\lambda(t)y+\sigma(t)]+\beta(t)y,\tag{63}$$

where $\lambda=\lambda(t)$, $\sigma=\sigma(t)$, and $\alpha=\alpha(t)$; the other functional coefficients are given by

$$a=\frac{a_0}{\lambda^3}\exp\left(\nu\int\lambda^2\,dt\right),\quad b=\frac{\lambda'_t}{\lambda},\quad c=\frac{\sigma'_t}{\lambda},$$
$$\beta=-\frac{1}{a}(\alpha'_t+2b\alpha-\nu\alpha\lambda^2),\quad s=\frac{s_0}{\lambda^2}\exp\left(\nu\int\lambda^2\,dt\right),\tag{64}$$

where a_0 and s_0 are arbitrary constants.

2°. System (25), (26) admits exact solutions of the form

$$f=a(t)\sinh[\lambda(t)y+\sigma(t)]+b(t)y+c(t),$$
$$h=\alpha(t)\sinh[\lambda(t)y+\sigma(t)]+s(t)\cosh[\lambda(t)y+\sigma(t)]+\beta(t)y,$$

where $\lambda=\lambda(t)$, $\sigma=\sigma(t)$, and $\alpha=\alpha(t)$ are arbitrary functions; the functional coefficients $a=a(t)$, $b=b(t)$, $c=c(t)$, $s=s(t)$, and $\beta=\beta(t)$ are given by (64).

5. Solutions to the general determining system

5.1. Solutions involving exponential functions

System (10)–(12) admits the following exact solution involving exponential functions:

$$f=k\psi e^{-z}+\beta,\quad g=-k\varphi e^{-z}+\delta,\quad \overline{h}=pe^{-z}+qz.\tag{65}$$

The seven time-dependent functional coefficients k, p, q, β, δ, φ, and ψ as well as the functional coefficient b, appearing in the system, are to be determined in the analysis. Substituting (65) into system (10)–(12) and collecting the coefficients of the different exponential functions as well as the free coefficient, we arrive at an underdetermined system of ordinary differential equations:

$$(a+c)'_t=0,\tag{66}$$

$$k\psi'_t+\psi k'_t+k\psi(s+b)+ak\varphi+k\beta\psi^2-k\delta\varphi\psi-\nu k\psi(\varphi^2+\psi^2)=0,\tag{67}$$

$$k\varphi'_t+\varphi k'_t+k\varphi(s-b)-ck\psi+k\beta\varphi\psi-k\delta\varphi^2-\nu k\varphi(\varphi^2+\psi^2)=0,\tag{68}$$

$$p'_t+[\beta\psi-\delta\varphi-\nu(\varphi^2+\psi^2)]p+k(q+\beta\varphi+\delta\psi)(\varphi^2+\psi^2)=0.\tag{69}$$

In view of the relations (9), it follows from Eq. (66) that

$$b=\frac{2C_1\varphi\psi+\psi\psi'_t-\varphi\varphi'_t}{\varphi^2+\psi^2},\tag{70}$$

where C_1 is an arbitrary constant.

Multiplying Eq. (67) by φ and Eq. (68) by $-\psi$ and adding together followed by rearranging with the aid of (9) and (70), we obtain

$$\varphi\psi'_t-\psi\varphi'_t+C_1(\varphi^2+\psi^2)=0.\tag{71}$$

In Eq. (71), which relates φ and ψ, we change to new variables:

$$\varphi=\rho\cos\xi,\quad \psi=-\rho\sin\xi,$$

where $\rho=\rho(t)$ and $\xi=\xi(t)$. As a result, we get the simple equation $\xi'_t=C_1$. It follows that

$$\varphi=\rho(t)\cos(C_1t+C_2),\quad \psi=-\rho(t)\sin(C_1t+C_2),\tag{72}$$

where $\rho=\rho(t)$ is an arbitrary function and C_2 is an arbitrary constant. Substituting (72) into (9) and (70) yields the coefficients a, b, and c:

$$a=C_1-\frac{\rho'_t}{\rho}\sin(2C_1t+2C_2),\quad c=C_1+\frac{\rho'_t}{\rho}\sin(2C_1t+2C_2),$$
$$b=\frac{\rho'_t}{\rho}[\sin^2(C_1t+C_2)-\cos^2(C_1t+C_2)]=-\frac{\rho'_t}{\rho}\cos(2C_1t+2C_2).\tag{73}$$

Further, it follows from Eqs. (68) and (69) in view of the formulas (72) and (73) and some rearrangements that

$$\delta=\frac{1}{\varphi}\left(\frac{k'_t}{k}+4\frac{\rho'_t}{\rho}-\nu\rho^2+\beta\psi\right),$$

$$q=\frac{1}{k\rho^2}[p(\nu\rho^2+\delta\varphi-\beta\psi)-p'_t]-\beta\varphi-\delta\psi.\tag{74}$$

Formulas (72)–(74) with arbitrary functions $\rho=\rho(t)$, $k=k(t)$, $\beta=\beta(t)$, and $p=p(t)$ and arbitrary constants C_1 and C_2 determine the general solution to system (66)–(69). The corresponding exact solution to system (10)–(12) is given by (65), which generates an exact solution of the form (5) to the stream function equation (3).

Remark 5. System (10), (11) admits an exact solution that involves different exponential functions:

$$f=\alpha_1e^z+\alpha_2e^{-z}+\beta z+\gamma,\quad g=\delta_1e^z+\delta_2e^{-z}+\mu z+\varepsilon,$$

with the eight time-dependent functional coefficients α_1, α_2, β, γ, δ_1, δ_2, μ, and ε connected by five equations

$$\mu\varphi-\beta\psi=0,$$
$$\alpha'_1+\alpha_1(s+b)-a\delta_1+\alpha_1(\beta-\gamma)\psi+(\alpha_1\varepsilon-\beta\delta_1)\varphi-\nu\alpha_1(\varphi^2+\psi^2)=0,$$
$$\alpha'_2+\alpha_2(s+b)-a\delta_2+\alpha_2(\beta+\gamma)\psi-(\alpha_2\varepsilon+\beta\delta_2)\varphi-\nu\alpha_2(\varphi^2+\psi^2)=0,$$
$$\delta'_1+\delta_1(s-b)+c\alpha_1+\delta_1(\varepsilon-\mu)\varphi+(\alpha_1\mu-\gamma\delta_1)\psi-\nu\delta_1(\varphi^2+\psi^2)=0,$$
$$\delta'_2+\delta_2(s-b)+c\alpha_2-\delta_2(\varepsilon+\mu)\varphi+(\alpha_2\mu+\gamma\delta_2)\psi-\nu\delta_2(\varphi^2+\psi^2)=0.$$

5.2. Solutions involving trigonometric functions

System (10), (11), (14) admits the following exact solution involving trigonometric functions:

$$f=k\psi\sin(z+\lambda)+\beta,\quad g=-k\varphi\sin(z+\lambda)+\delta,$$
$$\overline{h}=p\sin(z+\lambda)+q\cos(z+\lambda)+rz,\tag{75}$$

with the nine time-dependent functional coefficients k, p, q, r, λ, β, δ, φ, and ψ involved in the solution as well as the functional coefficient b appearing in the system to be determined in the analysis. By substituting (75) into system (10), (11), (14) and collecting the coefficients of the different trigonometric functions as well as the free term, we arrive at the underdetermined system of ordinary differential equations

$$(a+c)'_t=0,\tag{76}$$

$$\lambda'_t-\beta\psi+\delta\varphi=0,\tag{77}$$

$$(k\psi)'_t+k(s+b)\psi+ak\varphi+\nu k\psi(\varphi^2+\psi^2)=0,\tag{78}$$

$$(k\varphi)'_t+k(s-b)\varphi-ck\psi+\nu k\varphi(\varphi^2+\psi^2)=0,\tag{79}$$

$$q'_t+p(\lambda'_t-\beta\psi+\delta\varphi)+\nu q(\varphi^2+\psi^2)=0,\tag{80}$$

$$p'_t-q(\lambda'_t-\beta\psi+\delta\varphi)+\nu p(\varphi^2+\psi^2)+k(r+\beta\varphi+\delta\psi)(\varphi^2+\psi^2)=0,\tag{81}$$

whose four functional coefficients can be considered arbitrary. Recall that the functions a, c, and s are expressed in terms of b, φ, ψ through the formulas (9) and (13).

Eq. (76) coincides with (66) and results in the formula (70) for b.

By multiplying Eq. (78) by φ and Eq. (79) by $-\psi$ and adding together followed by some rearrangements with the aid of (9) and (70), we arrive at an equation coinciding with Eq. (71). Using similar arguments to those in Section 5.1, we arrive at the same formulas (72) and (73) for the functional coefficients φ, ψ, a, b, and c.

Substituting (72) and (73) into (78) and integrating, we obtain

$$k=\frac{C_3}{\rho^4(t)}\exp\left[-\nu\int\rho^2(t)\,dt\right],\tag{82}$$

where C_3 is an arbitrary constant.

Solving the remaining three equations (77), (80), and (81) yields

$$\lambda=C_4-\int\rho[\beta\sin(C_1t+C_2)+\delta\cos(C_1t+C_2)]\,dt,$$

$$q=C_5\exp\left(-\nu\int\rho^2\,dt\right),$$
$$r=-\beta\rho\cos(C_1t+C_2)+\delta\rho\sin(C_1t+C_2)-\frac{1}{k\rho^2}(p'_t+\nu p\rho^2),\tag{83}$$

where C_4 and C_5 are arbitrary constants.

To sum up, the formulas (72), (73), (82), and (83) define the general solution to system (76)–(81), which involves four arbitrary functions of time $\rho=\rho(t)$, $\beta=\beta(t)$, $\delta=\delta(t)$, and $p=p(t)$ as well as five arbitrary constants C_1, ..., C_5. The corresponding exact solution to system (10)–(12) is given by Eq. (65), which generates an exact solution of the form (5) to the stream function equation (3).

Remark 6. System (10), (11) admits exact solutions of the form

$$f=\alpha_1\cos z+\alpha_2\sin z+\beta z+\gamma,\quad g=\delta_1\cos z+\delta_2\sin z+\mu z+\varepsilon,$$

with the eight time-dependent functional coefficients α_1, α_2, β, γ, δ_1, δ_2, μ, and ε connected by five equations

$$\mu\varphi-\beta\psi=0,$$
$$\alpha'_1+\alpha_1(s+b)-a\delta_1+(\alpha_1\beta-\alpha_2\gamma)\psi+(\alpha_2\varepsilon-\beta\delta_1)\varphi+\nu\alpha_1(\varphi^2+\psi^2)=0,$$
$$\alpha'_2+\alpha_2(s+b)-a\delta_2+(\alpha_1\gamma+\alpha_2\beta)\psi-(\alpha_1\varepsilon+\beta\delta_2)\varphi+\nu\alpha_2(\varphi^2+\psi^2)=0,$$
$$\delta'_1+\delta_1(s-b)+c\alpha_1+(\delta_2\varepsilon-\mu\delta_1)\varphi+(\alpha_1\mu-\gamma\delta_2)\psi+\nu\delta_1(\varphi^2+\psi^2)=0,$$
$$\delta'_2+\delta_2(s-b)+c\alpha_2-(\delta_1\varepsilon+\delta_2\mu)\varphi+(\alpha_2\mu+\gamma\delta_1)\psi+\nu\delta_2(\varphi^2+\psi^2)=0.$$

5.3. *Mixed steady–unsteady solutions to the determining system*

A wide class of linearizing solutions to system (10)–(12) can be obtained by setting

$$\varphi=\text{const},\quad \psi=\text{const},\quad b=\text{const},\quad f=f(z),\quad g=g(z).$$

In this case, the first two equations, (10) and (11), represent a stationary system of ordinary differential equations for f and g. The third one is a linear partial differential equation with coefficients independent of t, which suggests that it can be analyzed using the Laplace or Fourier transform.

6. Reduction of the determining system to fewer equations. Order reduction of the determining system

6.1. *Reduction of system (10)–(12) to two equations*

Eq. (11) can be identically satisfied with

$$g=g_1(t)z+g_0(t),\quad b=\frac{\psi'_t}{\psi}-\varphi g_1(t),\tag{84}$$

where $g_0=g_0(t)$, $g_1=g_1(t)$, $\varphi=\varphi(t)$, and $\psi=\psi(t)$ are arbitrary functions, with the functional coefficients a and c defined by (9). In this case, Eq. (10) becomes isolated and then system (10)–(12) reduces to only two equations (omitted here).

6.2. *Reduction of subsystem (10), (11) to a single equation*

We will seek solutions to system (10), (11) of the form

$$f=\alpha(t)u(t,z)+\beta(t),\quad g=\gamma(t)u(t,z)+\delta(t),\tag{85}$$

with the functions $\alpha=\alpha(t)$, $\beta=\beta(t)$, $\gamma=\gamma(t)$, $\delta=\delta(t)$, and $u=u(t,z)$ to be determined. We require Eqs. (10) and (11) to coincide after substituting the expressions (85). This results in a single equation relating two functions of time:

$$\gamma\alpha'_t-\alpha\gamma'_t+2b\alpha\gamma-a\gamma^2-c\alpha^2=0.\tag{86}$$

In view of (86), we get the following equation for $u=u(t,z)$:

$$u_{tzz}+\lambda u_{zz}+(\delta\varphi-\beta\psi)u_{zzz}+(\alpha\psi-\gamma\varphi)(u_zu_{zz}-uu_{zzz})=\nu(\varphi^2+\psi^2)u_{zzzz}\tag{87}$$

with

$$\lambda=\frac{1}{\alpha}[\alpha'_t+\alpha(s+b)-a\gamma]=\frac{1}{\gamma}[\gamma'_t+\gamma(s-b)+c\alpha].\tag{88}$$

Substituting (85) into Eq. (12) yields

$$\overline{h}_{tzz}+(\alpha\psi-\gamma\varphi)u_{zz}\overline{h}_z+[(\gamma\varphi-\alpha\psi)u+\delta\varphi-\beta\psi]\overline{h}_{zzz}+Q[u]=\nu(\varphi^2+\psi^2)\overline{h}_{zzzz},\tag{89}$$

where

$$Q[u]=(a+c)'_t+2(\alpha\varphi+\gamma\psi)'_tu_z+2(\alpha\varphi+\gamma\psi)u_{tz}+[(3\alpha\delta-\beta\gamma)\varphi^2$$
$$+(\alpha\delta-3\beta\gamma)\psi^2+2(\gamma\delta-\alpha\beta)\varphi\psi]u_{zz}+2(\alpha\varphi+\gamma\psi)(\gamma\varphi-\alpha\psi)uu_{zz}$$
$$-4\nu(\varphi^2+\psi^2)(\alpha\varphi+\gamma\psi)u_{zzz}.\tag{90}$$

Eqs. (87)–(89) include seven time-dependent functional parameters b, α, β, γ, δ, φ, and ψ, which are constrained by a single equation (86). Hence, six parameters can be considered free. By varying the parameters appropriately, we can drastically simplify Eqs. (87)–(89). Let us consider two special cases.

Case 1: We set

$$\alpha=k\varphi,\quad \gamma=k\psi,\tag{91}$$

where $k=k(t)$ is an arbitrary function. In this case, Eq. (87) becomes linear; with the substitution $v=u_{zz}$, it can be reduced to a second-order parabolic equation and further to the classical heat equation (see Eq. 4 on page 147 in [74]). In addition, if the conditions (91) hold, Eq. (89) can also be reduced, with the change of variable $H=\overline{h}_{zz}$, to a second-order parabolic equation and further to the heat equation with source.

Case 2: We set

$$\alpha=k\psi,\quad \gamma=-k\varphi,\tag{92}$$

with $k=k(t)$ to be determined. If the conditions (92) hold, Eqs. (87) and (89) become

$$u_{tzz}+\lambda u_{zz}+(\delta\varphi-\beta\psi)u_{zzz}$$
$$+k(\varphi^2+\psi^2)(u_zu_{zz}-uu_{zzz})=\nu(\varphi^2+\psi^2)u_{zzzz},\tag{93}$$

$$\overline{h}_{tzz}+(a+c)'_t+k(\varphi^2+\psi^2)(\beta\varphi+\delta\psi)u_{zz}+(\delta\varphi-\beta\psi)\overline{h}_{zzz}$$
$$+k(\varphi^2+\psi^2)(\overline{h}_zu_{zz}-u\overline{h}_{zzz})=\nu(\varphi^2+\psi^2)\overline{h}_{zzzz},\tag{94}$$

where

$$\lambda=\frac{k'_t}{k}+\left(\frac{1}{\psi^2}+\frac{2}{\varphi^2+\psi^2}\right)(\varphi\varphi'_t+\psi\psi'_t)+\frac{b}{\psi^2}(\varphi^2+\psi^2).\tag{95}$$

In view of (9) and (13), relation (86), connecting the functional parameters, can be rewritten as

$$(\varphi^2-\psi^2)(\varphi\varphi'_t+\psi\psi'_t)+b(\varphi^2+\psi^2)^2=0.$$

6.3. *Reduction of system (10)–(12) to a single non-linear PDE*

The following theorem holds true, which allows one to find exact solutions to the three-equation system (10)–(12) using solutions to a single non-linear PDE.

Theorem 4. *Suppose the functional coefficients a, b, and c in (5) are defined by (73). Then system (10)–(12) admits the exact solution*

$$f=-k\rho\sin\xi\,u(t,z)+\beta,$$
$$g=-k\rho\cos\xi\,u(t,z)+\delta,$$
$$\overline{h}=pu(t,z)+qz+r,\tag{96}$$

where $k=k(t)$, $\rho=\rho(t)$, $\beta=\beta(t)$, $\delta=\delta(t)$, $p=p(t)$, and $r=r(t)$ are arbitrary functions,

$$\xi=C_1t+C_2,$$
$$q=\frac{1}{k\rho^2}[p\rho(\nu\rho+\delta\cos\xi+\beta\sin\xi)-p'_t]+\rho(\delta\sin\xi-\beta\cos\xi), \qquad (97)$$

with C_1 and C_2 being arbitrary constants; *the function $u=u(t,z)$ is a solution of the non-linear partial differential equation with variable coefficients*

$$u_{tzz}+\left(\frac{k'_t}{k}+4\frac{\rho'_t}{\rho}\right)u_{zz}+\rho(\delta\cos\xi+\beta\sin\xi)u_{zzz}+k\rho^2(u_zu_{zz}-uu_{zzz})=\nu\rho^2u_{zzzz}. \qquad (98)$$

By varying the arbitrary functions appearing in (96)–(98) appropriately, one can find exact solutions to system (10)–(12).

Example 6. We look for a generalized separable solution to Eq. (98) of the form

$$u=Az^m+Bz+C, \qquad (99)$$

with the functional coefficients $A=A(t)$, $B=B(t)$, and $C=C(t)$ and constant m to be determined. By substituting (99) into (98) and performing simple rearrangements, we obtain

$$m=-1,\quad A=\frac{6\nu}{k},\quad B=-\frac{\rho'_t}{k\rho^2},\quad C=\frac{1}{k\rho}(\delta\cos\xi+\beta\sin\xi). \qquad (100)$$

Formulas (96), (97), (99), and (100) define an exact solution to system (10)–(12) with coefficients (5). This solution involves five arbitrary functions of time: $k=k(t)$, $\rho=\rho(t)$, $\beta=\beta(t)$, $\delta=\delta(t)$, and $p=p(t)$.

6.4. *Order reduction of system (10)–(12)*

System (10), (11) admits order reduction. Indeed, let us integrate the system with respect to z to obtain

$$f_{tz}+(s+b)f_z-ag_z+\psi(f_z^2-ff_{zz})+\varphi(gf_{zz}-f_zg_z)=\nu(\varphi^2+\psi^2)f_{zzz}+p_0(t), \qquad (101)$$

$$g_{tz}+(s-b)g_z+cf_z+\psi(g_zf_z-fg_{zz})+\varphi(gg_{zz}-g_z^2)=\nu(\varphi^2+\psi^2)g_{zzz}+q_0(t), \qquad (102)$$

where $p_0(t)$ and $q_0(t)$ are arbitrary functions. The order of Eq. (12) can be reduced with the substitution $H=(\varphi^2+\psi^2)h_z$, which results in

$$\begin{aligned}&H_{tz}+(\psi f_{zz}-\varphi g_{zz})H+(\varphi g-\psi f)H_{zz}+a'_t+c'_t+2\varphi'_tf_z+2\psi'_tg_z+2\varphi f_{tz}\\&\quad+2\psi g_{tz}+[(3\varphi^2+\psi^2)g-2\varphi\psi f]f_{zz}+[2\varphi\psi g-(\varphi^2+3\psi^2)f]g_{zz}\\&\quad-4\nu(\varphi^2+\psi^2)(\varphi f_{zzz}+\psi g_{zzz})=\nu(\varphi^2+\psi^2)H_{zzz}.\end{aligned} \qquad (103)$$

Interestingly, system (101), (102) is more difficult to analyze for the purpose of finding exact solutions than the original system (10), (11), which contains higher-order derivatives. This is because the combinations $\Phi=f_z^2-ff_{zz}$ and $\Psi=g_zf_z-fg_{zz}$ are not reduced to zero with the simplest trigonometric or hyperbolic functions of z, unlike the combinations $\tilde\Phi=\Phi_z=f_zf_{zz}-ff_{zzz}$ and $\tilde\Psi=\Psi_z=g_zf_{zz}-fg_{zzz}$, appearing in the original system.

6.5. *An independent equation for a linear combination of f and g. Reduction of system (10)–(12) to a triangular form*

Let us multiply Eq. (10) by ψ and Eq. (11) by $-\varphi$ and add together. By introducing the new function

$$\Theta=\psi f-\varphi g, \qquad (104)$$

which is a linear combination of the desired functions f and g, taking into account the relations (8), and performing some rearrangements, we obtain the following independent equation for $\Theta=\Theta(t,z)$:

$$\Theta_{tzz}+s\Theta_{zz}+\Theta_z\Theta_{zz}-\Theta\Theta_{zzz}=\nu(\varphi^2+\psi^2)\Theta_{zzzz}. \qquad (105)$$

By expressing g from (104) via f and Θ, substituting it in Eq. (10), and using the first formula in (9), we arrive at the equation

$$f_{tzz}+\left(s-\frac{\varphi'_t}{\varphi}\right)f_{zz}+\frac{a}{\varphi}\Theta_{zz}+\Theta_{zz}f_z-\Theta f_{zzz}=\nu(\varphi^2+\psi^2)f_{zzzz}, \qquad (106)$$

which is linear in f.

The combination of the three equations (105), (106), and (12) represents a triangular system in the sense that the first equation depends on Θ alone, the second depends on f and Θ, and the third depends on h, f, and Θ. This system can be solved consecutively, starting from the first equation. In Eq. (12), the function g should be expressed in terms of f and Θ using (104). Any particular solution (e.g., a stationary solution) to Eq. (105) generates a linearizing solution to the triangular system.

7. Brief conclusions

We have studied unsteady Navier–Stokes equations with two space variables. The corresponding non-linear fourth-order PDE for the stream function w with three independent variables t, x, and y has been shown to admit functional separable solutions of the form

$$w=xf(t,z)+yg(t,z)+h(t,z)+\tfrac{1}{2}a(t)x^2+b(t)xy+\tfrac{1}{2}c(t)y^2,$$
$$z=\varphi(t)x+\psi(t)y,$$

with the functions $f(t,z)$, $g(t,z)$, and $h(t,z)$ described by a system of three partial differential equations with two independent variables. We have found a number of exact solutions to this system, which generate new classes of exact solutions to the unsteady Navier–Stokes equations. All solutions involve two or more arbitrary functions of a single argument as well as a few free parameters. Many of the solutions obtained are expressed in terms of elementary functions, provided that the arbitrary functions are also elementary (such solutions, having relatively simple form and presenting significant arbitrariness, can be especially useful for solving certain model problems and testing numerical and approximate analytical hydrodynamic methods). We have also obtained a few new solutions to the Navier–Stokes equations that are expressible in terms of solutions to linear PDEs. We have stated and proved a few theorems that allow one to construct and generalize exact solutions. We have presented several examples illustrating how the results obtained can be used to describe some model flows of viscous incompressible fluids, including a flow in a strip with permeable boundaries, flow in a strip with boundary extrusion, flow onto a shrinking plane, and others. We have discussed a few blow-up flows generated by solutions with singularities at finite times.

Appendix

The functional coefficients on the left-hand side of Eq. (6) are expressed as

$$\begin{aligned}A&=(\varphi^2+\psi^2)(\varphi'_t+b\varphi-a\psi)f_{zzz},\\B&=(\varphi^2+\psi^2)[(\psi'_t+c\varphi-b\psi)f_{zzz}+(\varphi'_t+b\varphi-a\psi)g_{zzz}],\\C&=(\varphi^2+\psi^2)(\psi'_t+c\varphi-b\psi)g_{zzz},\\D&=[(2\varphi^2+\psi^2)'_t+\psi(\varphi^2+\psi^2)f_z+b(3\varphi^2+\psi^2)-2a\varphi\psi]f_{zz}\\&\quad+[2\psi\varphi'_t-\varphi(\varphi^2+\psi^2)f_z-a(\varphi^2+3\psi^2)+2b\varphi\psi]g_{zz}\\&\quad+(\varphi^2+\psi^2)(\varphi g-\psi f)f_{zzz}+(\varphi^2+\psi^2)f_{tzz}-\nu(\varphi^2+\psi^2)^2f_{zzzz}\\&\quad+(\varphi^2+\psi^2)(\varphi'_t+b\varphi-a\psi)h_{zzz},\end{aligned}$$

$$
\begin{aligned}
E = {} & [(\varphi^2+2\psi^2)'_t - \varphi(\varphi^2+\psi^2)g_z - b(\varphi^2+3\psi^2) + 2c\varphi\psi]g_{zz} + [2\varphi\psi'_t \\
& + \psi(\varphi^2+\psi^2)g_z + c(3\varphi^2+\psi^2) - 2b\varphi\psi]f_{zz} + (\varphi^2+\psi^2)(\varphi g - \psi f)g_{zzz} \\
& + (\varphi^2+\psi^2)g_{tzz} - \nu(\varphi^2+\psi^2)^2 g_{zzzz} + (\varphi^2+\psi^2)(\psi'_t + c\varphi - b\psi)h_{zzz}, \\
F = {} & (\varphi^2+\psi^2)(\psi f_{zz} - \varphi g_{zz})h_z + (\varphi^2+\psi^2)'_t h_{zz} + (\varphi^2+\psi^2)(\varphi g - \psi f)h_{zzz} \\
& + (\varphi^2+\psi^2)h_{tzz} - \nu(\varphi^2+\psi^2)^2 h_{zzzz} + a'_t + c'_t + 2\varphi'_t f_z + 2\psi'_t g_z \\
& + [(3\varphi^2+\psi^2)g - 2\varphi\psi f]f_{zz} + [2\varphi\psi g - (\varphi^2+3\psi^2)f]g_{zz} \\
& + 2\varphi f_{tz} + 2\psi g_{tz} - 4\nu(\varphi^2+\psi^2)(\varphi f_{zzz} + \psi g_{zzz}).
\end{aligned}
$$

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