THE IMAGES OF LIE POLYNOMIALS EVALUATED ON 2×2 MATRICES OVER AN ALGEBRAICALLY CLOSED FIELD.

ALEXEI KANEL-BELOV, SERGEY MALEV, LOUIS ROWEN

Abstract. Kaplansky asked about the possible images of a polynomial f in several noncommuting variables. In this note we consider the case of f a Lie polynomial with constant term 0, and coefficients in an algebraically closed field K. We describe all the possible images of f in $M_2(K)$ and provide an example of f whose image is the set of trace zero matrices without nilpotent nonzero matrices. We provide an arithmetic criterion for this case.

1. Introduction

A **Lie polynomial** is an element of the free Lie algebra in the alphabet $\{x_i : i \in I\}$, cf. [Ra, p. 8]. Intuitively, a Lie polynomial is a sum of Lie monomials αh, where h is a Lie word, built inductively: each x_i is a Lie word of degree 1, and if h_1, h_2 are Lie words of degree d_1 and d_2, then $[h_1, h_2]$ is a Lie word of degree $d_1 + d_2$.

This note consists of two parts. In the first part we describe the motivation. In the second part we classify the possible images of Lie polynomials evaluated on 2×2 matrices and consider the 3×3 case. This note is the continuation of [BeMR1], in which we considered the question, reputedly raised by Kaplansky, of the possible image set $\operatorname{Im} f$ of a polynomial f on matrices. See [BeMR1] for the historical background.

In [BeMR1] the field K was required to be quadratically closed. In [M] results were provided over real closed field and arbitrary fields.

Here we are interested in images of Lie polynomials on matrices. Since $[f, g]$ can be interpreted as $fg - gf$, in this way we can identify any Lie polynomial with an associative polynomial; hence, any set that can arise as the image of a Lie polynomial also fits into the framework of the associative theory, so our challenge here is to find examples of Lie polynomials which achieve the sets described in [BeMR1, BeMR2, BeMR3]. As we shall see, this task is not as easy as it may seem at first glance.

1.1. A Group theoretical problem and its relation with the Lie theoretical problem.

Let w be an element of the free group of m letters $x_1, x_2, \dots x_{m-1}$ and x_m. Given a group G, we consider the map $f_{w,G} : G^m \to G$ corresponding to the word w. This map is called a *word map*, which for convenience we also notate as w instead of $f_{w,G}$. There is a group conjecture (see [BeKP, Question 2] for the more general case):

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Conjecture 1. *If the field K is algebraically closed of characteristic* 0*, then the image of any nontrivial group word $w(x_1, \dots, x_m)$ on* $\mathrm{PSL}_2(K)$ *is* $\mathrm{PSL}_2(K)$.

Remark 1. Note that if one takes the group SL_2 instead of PSL_2, Conjecture 1 fails, since the matrix $-I + e_{12}$ does not belong to the image of the word map $w = x^2$.

Example 1. When $\mathrm{Char}\, K = p > 0$, the image of the word map $w(x) = x^p$ evaluated on $\mathrm{PSL}_2(K)$ is not $\mathrm{PSL}_2(K)$. Indeed, otherwise the matrix $I + e_{12}$ could be written as x^p for $x \in \mathrm{PSL}_2(K)$. If the eigenvalues of x are equal, then $x = I + n$ where n is nilpotent. Therefore $x^p = (I + n)^p = I + pn = I$. If the eigenvalues of x are not equal, then x is diagonalizable and therefore x^p is also diagonalizable, a contradiction.

Remark 2. There is a connection between questions related to matrix groups and Lie theoretical questions. Consider the group generated by matrices of the type $g_i = I + a_i$, where a_i are generic matrices with trace 0. Let w be the word map, then $w(g_1, \dots, g_s) = I + g + R$, where $g \neq 0$ is the sum of all terms of minimal nonzero degree. Therefore (according to [Zu] for 2×2 matrices over field of characteristic not 2, and [Ze] for $n \times n$ matrices where $\mathrm{Char}\, K = p > p(n)$), g is a Lie polynomial. This fact lets us show that a free pro-p group cannot be embedded to the group of $n \times n$ matrices if $p \gg n$ (if $n = 2$ and $p > 2$ it is proved by Zubkov in [Zu]). Hence Lie algebraical problems play an important role for investigation of the possible images of word maps.

Lemma 1 (Liebeck, Nikolov, Shalev, cf. also [G] and [Ban])**.** $\mathrm{Im}\, w$ *contains all matrices from* $\mathrm{PSL}_2(K)$ *which are not unipotent.*

Proof. According to [Bo] the image of the word map w must be Zariski dense in $\mathrm{SL}_2(K)$. Therefore the image of $\mathrm{tr}\, w$ must be Zariski dense in K. Note that $\mathrm{tr}\, w$ is a homogeneous rational function and K is algebraically closed. Hence, $\mathrm{Im}\,(\mathrm{tr}\, w) = K$. For any $\lambda \neq \pm 1$ any matrix with eigenvalues λ and λ^{-1} belongs to the image of w since there is a matrix with trace $\lambda + \lambda^{-1}$ in $\mathrm{Im}\, w$ and any two matrices from SL_2 with equal trace (except trace ± 2) are similar. Note that the identity matrix I belongs to the image of any word map. $\square$

However the question whether one of the matrices $(I + e_{12})$ or $(-I - e_{12})$ (which are equal in PSL_2) belongs to the image of w remains open. We conjecture that $I + e_{12}$ must belong to $\mathrm{Im}\, w$. Note that if there exists i such that the degree of x_i in w is $k \neq 0$ then we can consider all $x_j = I$ for $j \neq i$ and $x_i = I + e_{12}$. Then the value of w is $(I + e_{12})^k = I + ke_{12}$ and this is a unipotent matrix since $\mathrm{Char}\, K = 0$, and thus $\mathrm{Im}\, w = \mathrm{PSL}_2(K)$. Therefore it is interesting to consider word maps $w(x_1, \dots, x_m)$ such that the degree of each x_i is zero.

This is why Conjecture 1 can be reformulated as follows:

Conjecture 2. *Let $w(x_1, \dots, x_m)$ be a group word whose degree ni each x_i is* 0*. Then the image of w on G must be* $\mathrm{PSL}_2(K)$*, where $G = \mathrm{GL}_2(K)/\{\pm 1\}$.*

One can consider matrices $z_i = \frac{x_i}{\sqrt{\det x_i}}$ and note that $w(z_1, \dots, z_m) = w(x_1, \dots, x_m)$.

For Conjecture 2 we take $y_i = x_i - I$. Then we can open the brackets in

$$w(1 + y_1, 1 + y_2, \dots, 1 + y_m) = 1 + f(y_1, \dots, y_m) + g(y_1, \dots, y_m),$$

where f is a homogeneous Lie polynomial of degree d, and g is the sum of terms of degree greater than d. Therefore it is interesting to investigate the possible images of Lie polynomials, whether it is possible that the image of l does not contain nilpotent matrices. Unfortunately it is possible. More general questions about surjectivity of word maps in groups and polynomials in algebras are considered in [BeKP].

2. The images of homogeneous Lie polynomials on $M_n(K)$ and sl_n

As mentioned in the introduction, the situation for Lie polynomials is considerably more intricate than for regular polynomials, for the simple reason that the most prominent polynomials in the theory, the standard polynomial s_n and the Capelli polynomial c_n are not obviously Lie polynomials. Even the case where a Lie polynomial takes on only zero values, i.e., is a PI, is nontrivial, although it has been studied in two important books [Bak, Ra].

In order to pass to the associative theory, we make use of the **adjoint algebra** $\mathrm{ad}L = \{\mathrm{ad}_a : L \to L : a \in L\}$ given by $\mathrm{ad}_a(b) = [a, b]$. Note that

$$\dim_K \mathrm{ad}L < \dim \mathrm{End}_K L = (\dim_K L)^2.$$

Also, it is well-known that the map $a \mapsto \mathrm{ad}_a$ defines a Lie algebra homomorphism $L \to \mathrm{ad}L$.

Remark 3.

$$\mathrm{ad}_{a_1} \dots \mathrm{ad}_{a_n}(a) = [a_1, \dots, [a_{n-1}, [a_n, a]] \dots].$$

In this way, any "ad"-monomial corresponds to a Lie monomial, and thus any "ad"-polynomial $f(\mathrm{ad}_{x_1}, \dots, \mathrm{ad}_{x_n})$ gives rise to a Lie polynomial $f(x_1, \dots, x_n, x_{n+1})$ taking on the same values, and in which x_{n+1} appears of degree 1 in each Lie monomial, in the innermost set of Lie brackets.

Conversely, we have:

Proposition 1. *Suppose $f(x_1, \dots, x_n, x_{n+1})$ is a Lie polynomial in which x_{n+1} appears in degree 1 in each Lie monomial. Then f corresponds to an ad-polynomial taking on the same values on L as f.*

Proof. In view of Remark 3, it suffices to show that any Lie monomial h can be rewritten in the free Lie algebra as a sum of Lie monomials in which x_{n+1} appears (in degree 1) in the innermost set of Lie brackets. This could be done directly by means of the Jacobi identity, but here is a slicker argument.

Write $h = [h_1, h_2]$, and we appeal to induction on the degree of h. If x_{n+1} appears say in h_2. If $h_2 = x_{n+1}$ then we are done since $h = [h_1, x_{n+1}]$ corresponds to ad_{h_1}. In general, by induction, h_2 corresponds to $\mathrm{ad}_{h_3}(x_{n+1})$, so $[h_1, h_2] = \mathrm{ad}_{h_1}(\mathrm{ad}_{h_3}(x_{n+1}))$ corresponds to $\mathrm{ad}_{h_1}\mathrm{ad}_{h_3}(x_{n+1}) = \mathrm{ad}_{[h_1,h_3](x_{n+1})}$, as desired. □

Example 2. For any Lie algebra L of degree n, let $t = 2(n^2 - 1)$ and take $f = s_t(\mathrm{ad}_{x_1}, \mathrm{ad}_{x_2}, \dots, \mathrm{ad}_{x_t})(x_{t+1})$. considering the ad_{x_l} as linear operators from sl_n Then s_t is a PI of $\mathrm{ad}L$, via the Amitsur-Levitzki theorem. Therefore, f is a consequence of the 0-operator on x_{t+1} and equals 0, and is a multilinear lie identity of L.

The same argument works for any Lie algebra L of dimension n. Then

$$f = c_t(\mathrm{ad}_{x_1}, \mathrm{ad}_{x_2}, \dots, \mathrm{ad}_{x_t})(x_{t+1}),$$

where $t = (n^2 - 1)^2$ is a Lie identity.

This gives rise to the following question:

Question 1. *What is the minimal degree of a Lie identity of* sl_n*?*

Even the answer for $n = 2$, given in [Ra, Theorem 36.1], is difficult.

Proposition 2. *s_n and c_n cannot be written as Lie polynomials for n odd.*

Proof. It is enough to show this for s_n, since it is a specialization of c_n. In view of Proposition 1 it is enough to find some matrix specialization of s_n which is nonzero on $M_n(F)$ when we specialize x_1 to a scalar matrix. But this is clear: For n odd we specialize $x_i \mapsto e_{i-1,i}$ for $2 \leq i \leq n$; then

$$s_n(I, e_{1,2}, \dots, e_{n-1,n}) = ne_{1,n}.$$

□

Example 3. (i) s_2 itself is a Lie polynomial.

(ii) s_4 is not a Lie polynomial. Indeed, it vanishes on sl_2, which has dimension 3, but every Lie identity of sl_2 has degree ≥ 5, by [Ra, Theorem 36.1].

Here is a computational proof. We have 15 multilinear Lie monomials of degree 4, namely $\frac{1}{2}\binom{4}{2} = 3$ of the form

$$[[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]] \tag{1}$$

and $2\binom{4}{2} = 12$ of the form

$$[[[x_{i_1}, x_{i_2}], x_{i_3}], x_{i_4}]. \tag{2}$$

But

$$[[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]] = \mathrm{ad}_{x_{i_2}} \mathrm{ad}_{[x_{i_3}, x_{i_4}]}(x_{i_1}) = (\mathrm{ad}_{x_{i_2}} \mathrm{ad}_{x_{i_3}}(x_{i_1}) - \mathrm{ad}_{x_{i_2}} \mathrm{ad}_{x_{i_4}}(x_{i_1})),$$

so we can rewrite the equations (1) in terms of (2). Furthermore, (2) can be reduced to eight Lie monomials, by means of the Jacobi identity.

Vishne found another dependence, reducing (2) to be spanned by seven Lie monomials, and with the help of Maple showed that they are independent.

One can also list these has $h_1, \dots, h_7$ and look for α_i such that $\sum \alpha_i h_i = \gamma s_4$ for some γ. This gives 24 equations in 7 variables, and one can check that the coefficient matrix has rank 7, thereby yielding no nontrivial solutions.

2.1. **The case $n = 2$.** The identities of sl_2 are determined in [Ra, Theorem 36.1]. We can provide a Lie identity of degree 5:

Theorem 1. *The multilinear Lie polynomial $p(x_1, \dots, x_5) = s_4(\mathrm{ad}_{x_1}, \mathrm{ad}_{x_2}, \mathrm{ad}_{x_3}, \mathrm{ad}_{x_4})(x_5)$ is PI.*

Proof. The set $\{\mathrm{ad}_x,\ x \in M_2(K)\}$ is a 3-dimensional linear space, therefore for any x_i the set $\{\mathrm{ad}_{x_1}, \mathrm{ad}_{x_2}, \mathrm{ad}_{x_3}, \mathrm{ad}_{x_4}\}$ is linearly dependent. Without loss of generality, $\mathrm{ad}_{x_4} = c_1 \mathrm{ad}_{x_1} + c_2 \mathrm{ad}_{x_2} + c_3 \mathrm{ad}_{x_3}$ for some $c_i \in K$. Note that p is multilinear, therefore

$$p(x_1, \dots, x_5) = \sum_{i=1}^{3} c_i p(x_1, x_2, x_3, x_i, x_5),$$

each term equals 0, therefore p is PI. □

According to [Š, Proposition 7.5], if p is a Lie polynomial of degree no more than 4 then $\mathrm{Im}\, p = \mathrm{sl}_2$. Therefore minimal degree of Lie PI is 5.

Theorem 2. *If f is a homogeneous Lie polynomial evaluated on the matrix ring $M_2(K)$ (where K is an algebraically closed field), then* $\operatorname{Im} f$ *is either* $\{0\}$*, or* K *(the set of scalar matrices), or the set of all non-nilpotent matrices having trace zero, or* $\mathrm{sl}_2(K)$*, or* $M_2(K)$*.*

Remark 4. The case of scalar matrices in Theorem 2 is possible only if $\operatorname{Char} K = 2$, and the last case $M_2(K)$ is possible only if $\deg f = 1$.

Proof of Theorem 2 According to the [BeMR1, Theorem 1] the image of f must be either $\{0\}$, or K, or the set of all non-nilpotent matrices having trace zero, or $\mathrm{sl}_2(K)$, or a dense subset of $M_2(K)$ (with respect to Zariski topology). Note that if at least one matrix having nonzero trace belongs to the image of f then $\deg f = 1$ and thus $\operatorname{Im} f = M_2(K)$. Therefore $\operatorname{Im} f$ is either $\{0\}$, or K (the set of scalar matrices), or $\mathrm{sl}_2(K)$, or $M_2(K)$. □

Theorem 3. *For K any algebraically closed field of characteristic $\neq 2$, the image of any Lie polynomial f (not necessarily homogeneous) evaluated on* $\mathrm{sl}_2(K)$ *is either* $\mathrm{sl}_2(K)$*, or* $\{0\}$*, or the set of trace zero non-nilpotent matrices.*

Proof. For f not a PI, we can write $f = f_k + f_{k+1} + \cdots + f_d$, where each f_i is a homogeneous Lie polynomial of degree k, and f_d is not PI. Therefore for any $c \in K$ we have

$$f(cx_1, cx_2, \ldots, cx_m) = c^k f_k + \cdots + c^d f_d.$$

Since f_d is not PI, there exist specializations of $x_1, \ldots, x_m$ for which $\det(f_d) \neq 0$. Fixing these specializations of the x_i, we consider $\det(c^k f_k + \cdots + c^d f_d)$ as a polynomial in c of degree $k + \cdots + d$. Since the leading coefficient is not zero and K is algebraically closed, its image is K. Thus for any $l \in K$ there exist $x_1, \ldots, x_m$ such that $\det(f) = l$. Hence (for $\operatorname{Char} K \neq 2$) any matrix with eigenvalues λ and $-\lambda$ for $\lambda \neq 0$ belongs to $\operatorname{Im} f$. Therefore $\operatorname{Im} f$ is either sl_2 or the set of trace zero non-nilpotent matrices.

□

Let us give examples of Lie polynomials with such images:

Examples. If $\operatorname{Char} K = 2$, then the case K also is possible: We take $f(x, y, z, t) = [[x, y], [z, t]]$. Any value of f is the Lie product of two trace zero matrices $s_1 = [x, y]$ and $s_2 = [z, t]$. Both can be written as $s_i = h_i + u_i + v_i$, where the h_i are diagonal trace zero matrices (which are scalar since $\operatorname{Char} K = 2$), the u_i are proportional to e_{12}, and the v_i are proportional to e_{21}. Thus $[s_1, s_2] = [u_1, v_2] + [u_2, v_1]$ is scalar.

Over an arbitrary field, $\operatorname{Im} f$ can indeed equal to $\{0\}$, or K, or the set of all non-nilpotent matrices having trace zero, or $\mathrm{sl}_2(K)$, or $M_2(K)$.

$\operatorname{Im} f = M_2(K)$ for any Lie polynomial of degree 1.

The image of $f(x, y) = [x, y]$ is sl_2.

Next, we construct a Lie polynomial whose image evaluated on $\mathrm{sl}_2(K)$ is the set of all non-nilpotent matrices having trace zero. We take the multilinear polynomial $h(u_1, \ldots, u_8)$ constructed in [DK] by Drensky and Kasparian which is central on 3×3 matrices. For the 2×2 matrices $x_1, \ldots, x_9$ we consider the homogeneous Lie polynomial

$$f(x_1, \ldots, x_9) = h(\mathrm{ad}_{[[[x_1,x_9],x_9],x_9]}, \mathrm{ad}_{x_2}, \mathrm{ad}_{x_3}, \ldots, \mathrm{ad}_{x_8})(x_9).$$

For any 2×2 matrix x the adjoint operator $\mathrm{ad}_x : \mathrm{sl}_2(K) \to \mathrm{sl}_2(K)$ that sends any trace zero matrix y to $[x, y]$. Since sl_2 is 3-dimensional, ad_x is a 3×3 matrix; hence, for any values of x_i, the value of f has to be proportional to x_9. However for nilpotent x_9 it has to be zero because $[[[x, n], n], n] = 0$ for any $x \in \mathrm{sl}_2(K)$ if n is nilpotent. (When we open the brackets we have the sum of 8 terms and each terms equals $n^k x n^{3-k}$ and for any integer k either $k \geq 2$ or $3 - k \geq 2$.) Thus the image of f is exactly the set of non-nilpotent trace zero matrices.

Another example of a homogeneous Lie polynomial with no nilpotent values is $f(x, y) = [[[x, y], x], [[x, y], y]]$ (see [BGKP, Example 4.9] for details).

Let $K[\xi]$ denote the polynomial algebra in infinitely many indeterminates (over K), a principal ideal domain. Let $K\{Y\}_n$ denote the algebra of generic matrices $y_k = (\xi^k_{i,j})$, where each $\xi^k_{i,j}$ denotes a distinct indeterminate [Row].

Remark 5. Our next theorem describes the situation in which the trace vanishing polynomial does not take on nonzero nilpotent values. It implies that any nontrivial word map w evaluated on PSL_2 is not surjective iff its projection to sl_2 given by $\mathrm{sl}_2 :\ x \mapsto x - \frac{1}{2}\mathrm{tr}\, x$ is a multiple of any prime divisor of $\det(\pi(w))$. This might help in answering Conjecture 1.

Theorem 4. *Let $f(x_1, \dots, x_m)$ be a trace vanishing polynomial, evaluated on $M_n(K[\xi])$. Let $\bar{f} = f(y_1, \dots, y_m)$. Then f takes on no nonzero nilpotent values on any integral domain containing K, iff each prime divisor d of $\det(\bar{f})$ also divides each entry of $\bar{f}$.*

Proof. ($\Rightarrow$) If some prime divisor d of $\det(\bar{f})$ does not divides $\bar{f}$, then $\bar{f}$ does not specialize to 0 modulo d. Therefore we have a nonzero matrix in the image of f which has determinant zero and also trace zero, and thus is nilpotent, a contradiction.

($\Leftarrow$) Assume that f takes on a nonzero nilpotent value over some extension integral domain of K. Thus $\det \bar{f}$ goes to 0 under the corresponding specialization of the $\xi^k_{i,j}$, so some prime divisor d of $\det(\bar{f})$ goes to 0, and $\bar{f}$ is not divisible by d. □

2.2. **The case $n = 3$.** Many new questions arise concerning the possible evaluation of Lie polynomials. In the associative case, the fact that the generic division algebra has a 3-central element implies that there is a homogeneous 3-central polynomial f for $M_3(K)$, i.e., all of whose values take on eigenvalues $c, \omega c, c\omega^2$, where ω is a cube root of 1. But any matrix with these eigenvalues is either scalar or has trace 0. This leads us to the basic question:

Question 2. *Is there a Lie polynomial f whose values on sl_3 all take on eigenvalues $c, \omega c, c\omega^2$, where ω is a primitive cube root of 1?*

According to [BeMR2, Theorem 3], if p is homogeneous polynomial with trace vanishing image, then $\mathrm{Im}\, p$ is one of the following:

- $\{0\}$,
- the set of scalar matrices (which can occur only if $\mathrm{Char}\, K = 3$),
- a dense subset of $\mathrm{sl}_3(K)$, or
- the set of 3-scalar matrices, i.e., the set of matrices with eigenvalues $(c, c\omega, c\omega^2)$, where ω is our cube root of 1.

We can give examples of Lie polynomials for $\{0\}$ and a dense subset of $\mathrm{sl}_3(K)$. The questions whether exists 3-central Lie polynomial and central (where K is a field of characteristic 3) remain being open.

Proposition 3. *Let $L = \mathrm{adsl}_2$, viewed as a subalgebra of* sl_3 *. The possible evaluations of an associative polynomial on L are:*

References

[Bak] Bakhturin, Yu. A. Identical relations in Lie algebras. VNU Science Press, Utrecht, 1987.

[Ban] Bandman, T., *Surjectivity of certain word maps on* $\mathrm{PSL}(2,\mathbb{C})$ *and* $\mathrm{SL}(2,\mathbb{C})$, arXiv:1407.3447

[BGKP] Bandman, T.; Gordeev, N.; Kunyavskii, B.; Plotkin E. *Equations in simple Lie algebras*, Journal of Algebra **355** (2012), 67–79.

[BeKP] Belov, A.; Kunyavskii, B.; Plotkin, E. *Word equations in simple groups and polynomial equations in simple algebras* Vestnik St. Petersburg University: Mathematics Surv.**46** (2013), 3–13.

[BeMR1] Belov, A.; Malev, S.; Rowen, L. *The images of non-commutative polynomials evaluated on* 2×2 *matrices*, Proc. Amer. Math. Soc **140** (2012), 465–478.

[BeMR2] Belov, A.; Malev, S.; Rowen, L. *The images of multilinear polynomials evaluated on* 3×3 *matrices*, submitted to the Proc. Amer. Math. Soc. (2013).

[BeMR3] Belov, A.; Malev, S.; Rowen, L., *Power-central polynomials on matrices,* submitted to the Journal of Pure and Applied Algebra (2013).

[BeR] Belov, A.; Rowen, L.H. *Computational Aspects of Polynomial Identities*, A. K. Peters Ltd., Wellesley, MA. (2005).

[Bo] Borel, A., *On free subgroups of semisimple groups*, Enseign. Math. **29** (1983), 151–164.

[Ch] Chuang, C.-L. *On ranges of polynomials in finite matrix rings*, Proceeding of the American Mathematical Society **110** (1990), no. 2, 293–302.

[DK] Drensky, V.; Kasparian, A. *A new central polynomial for* 3×3 *matrices*, Commun. Algebra **13** (1985), 745–752.

[G] Gordienko, A. S., Lecture notes on polynomial identities, preprint (2015)

[G] Gordeev, N. *On Engel words on simple algebraic groups.*, J. Algebra **425** (2015), 215–244.

[Ku1] Kulyamin, V. *Images of graded polynomials in matrix rings over finite group algebras* Russ. Math. Surv.**55** (2000), 345–346.

[Ku2] Kulyamin, V. *On images of polynomials in finite matrix rings, Thes. Cand. Phys.-Math. Sci., Moscow Lomonosov state University* Moscow (2000).

[L] Larsen, M. *Word maps have large image.* Israel J. Math. **139** (2004), 149-156.

[M] Malev, S. *The images of non-commutative polynomials evaluated on* 2×2 *matrices over an arbitrary field.* J. Algebra Appl. **13** (2014), no. 6.

[P] Procesi, C. *The invariant theory of* $n \times n$ *matrices*, Advances in Math. **19** (1976), 306–381.

[Ra] Razmyslov, Yu. P., Identities of Algebras and Their Representations, Nauka, Moscow (1989), AMS Translations of mathematical monographs **138** (1994).

[Row] Rowen, L. *Polynomial identities in ring theory*, Academic press, New York (1980).

[Š] Špenko, S. *On the image of a noncommutative polynomial*, Journal of Algebra **377** (2013), 298–311.

[Ze] Zelmanov, E. *Infinite algebras and pro-p-groups*, Progress in Mathematics **248** (2005), 403–413.

[Zu] Zubkov, A. *Non-abelian pro-p-groups cannot be represented by* 2×2 *matrices*, Siberian Math. J. **2** (1987), 742–747.

Department of Mathematics, Bar Ilan University, Ramat Gan, Israel
E-mail address: beloval@math.biu.ac.il

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel
E-mail address: malevs@math.huji.ac.il

Department of Mathematics, Bar Ilan University, Ramat Gan, Israel
E-mail address: rowen@math.biu.ac.il