Controlled dynamics in multicriteria optimization*

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A mathematical model of terminal control with two basic components: a controlled dynamics and a boundary value problem in the form of multicriteria equilibrium model, is considered. The boundary value problem describes a controlled object situated in a equilibrium state. Under the influence of external disturbances the object loses its state of stability and must be returned to equilibrium. The saddle point approach was used to do this, and the extraproximal method was applied to find a solution. The convergence of the method to solution was proved.

Boundary value problem. A group of m participating countries creates a community for the realization of some economic project. It is assumed that by the time of the community creation, the member countries have already identified their interests and objectives in the project, set types and amount of resources required to participate in integration. Interests of each of the participants are described by cost objective functions $f_i(x_1)$, $i = \overline{1,m}$, which are defined on a common set of resources $X_1 \subseteq \mathrm{R}^n$. Each of participants wants to minimize the cost of its contribution to the overall project. In the first approximation, this situation can be described as a simple multicriteria optimization problem:

$$f(x_1^*) \in \mathrm{ParetoMin}\{f(x_1) \mid x_1 \in X_1\}, \tag{1}$$

where $f(x_1) = (f_1(x_1), f_2(x_1), ..., f_m(x_1))$ is a vector criterion; convex scalar function $f_i(x_1)$ is value of resources that must be entered in the community by i-th participant to implement the project. The problem (1) generates a set of solutions in the form of vast variety of Pareto-optimal points.

Along with the individual interests of participants there exist also group interests, for example, the cost of the whole project. For different Pareto-optimal estimates this cost is different. It is natural to choose the project with a minimum value. Thus, it is necessary to formulate a mathematical model that takes into account both the individual interests of each participant and group (collective) interests of the community. As a result, the following two-person game with Nash equilibrium was

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proposed [1]:

$$\langle \lambda^*, f(x_1^*) \rangle \in \mathrm{Min}\{\langle \lambda^*, f(x_1) \rangle \mid x_1 \in X_1\}, \tag{2}$$

$$\langle \lambda - \lambda^*, f(x_1^*) - \lambda^* \rangle \leq 0, \quad \lambda \geq 0. \tag{3}$$

Formulation of terminal control problem. We add a controlled dynamics to the problem (2),(3) and formulate the following common dynamic model with multicriteria optimization boundary value problem:

$$\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \;\; t_0 \leq t \leq t_1, \; x(t_0) = x_0, \tag{4}$$

$$x(t_1) = x_1^* \in X_1 \subseteq \mathrm{R}^n, \; u(\cdot) \in \mathrm{U}, \tag{5}$$

$$\mathrm{U} = \{u(\cdot) \in \mathrm{L}_2^r[t_0, t_1] \mid \|u(\cdot)\|^2_{\mathrm{L}_2^r} \leq \mathrm{C}\}, \tag{6}$$

where x_1^* is x_1-component of solution for multicriteria equilibrium problem (2),(3). Here $D(t), B(t)$ are continuous matrices, x_0 is initial condition, $x(t) \in \mathrm{AC}_2^n[t_0, t_1]$ (linear variety of absolutely continuous functions). The dynamic model (2)-(6) describes the transition of controlled object from the initial state x_0 to a terminal state $x(t_1) = x_1^*$, which is given implicitly as the solution of (2),(3). We look for a control $u^*(t) \in \mathrm{U}$ such that the trajectory $x^*(t)$ has got by its right end to the appropriate component $x^*(t_1)$ of boundary value problem's solution.

Saddle point approach to the problem. We associate the problem (2)-(6) with the saddle-point-type function, which will play a role similar to the Lagrange function in convex programming:

$$\mathcal{L}(\lambda, \psi(t); x_1, x(t), u(t)) =$$

$$= \langle \lambda, f(x_1) - \frac{1}{2}\lambda \rangle + \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t) \rangle dt, \tag{7}$$

defined for all $(\lambda, \psi(\cdot)) \in \mathrm{R}^m_+ \times \Psi_2^n[t_0, t_1]$, $(x_1, x(t), u(t)) \in X_1 \times \mathrm{AC}^n[t_0, t_1]$ $\times \mathrm{U}$. In the case of regular constraints, the function (7) always has a saddle point $(\lambda_1^*, \psi^*(\cdot); x_1^*, x^*(\cdot), u^*(\cdot))$, which is the solution of the problem. Therefore, the problem (2)-(6) is reduced to finding the saddle points of (7).

Method to solve the problem. The dual extraproximal method that guarantees the convergence to the solution of saddle point problem (2)-(6), has been applied [1]:

$$\bar{\lambda}^k = \mathrm{argmin}\left\{\frac{1}{2}|\lambda - \lambda^k|^2 - \alpha\langle \lambda, f(x_1^k) - \frac{1}{2}\lambda \rangle \mid \lambda \geq 0\right\}, \tag{8}$$

$$\bar{\psi}^k(t) = \psi^k(t) + \alpha \left(D(t)x^k(t) + B(t)u^k(t) - \frac{d}{dt}x^k(t) \right), \qquad (9)$$

$$(x_1^{k+1}, x^{k+1}(\cdot), u^{k+1}(\cdot)) = \text{argmin} \left\{ \frac{1}{2}|x_1 - x_1^k|^2 + \right.$$

$$+\alpha\langle\bar{\lambda}^k, f(x_1) - \frac{1}{2}\bar{\lambda}^k\rangle + \frac{1}{2}\|x(t) - x^k(t)\|^2 + \frac{1}{2}\|u(t) - u^k(t)\|^2 +$$

$$\left. + \alpha \int_{t_0}^{t_1} \langle\bar{\psi}^k(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt \right\}, \qquad (10)$$

$$\lambda^{k+1} = \text{argmin} \left\{ \frac{1}{2}|\lambda - \lambda^k|^2 - \alpha\langle\lambda, f(x_1^{k+1}) - \frac{1}{2}\lambda\rangle \mid \lambda \geq 0 \right\}, \qquad (11)$$

$$\psi^{k+1}(t) = \psi^k(t) + \alpha \left(D(t)x^{k+1}(t) + B(t)u^{k+1}(t) - \frac{d}{dt}x^{k+1}(t) \right), \ \alpha > 0, \qquad (12)$$

where a minimum in (13) is computed in all $(x_1, x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$. A similar approach was considered in [2].

Theorem (on convergence of the method). *If the solution of equilibrium problem* (2)-(6) *exists, functions* $f_i(x_1)$, $i = \overline{1, m}$, *are convex and subject to Lipschitz condition with constant* L, *then the sequence generated by the dual extraproximal method* (8)-(12) *with the parameter* α, *satisfying the condition* $0 < \alpha < \alpha_0$, *where* α_0 *is a defined constant, contains a subsequence that converges to one of the solutions* $(\lambda^*, \psi^*(\cdot); x_1^*, x^*(\cdot), u^*(\cdot))$ *of the problem. In this case, the convergence in controls is weak, the convergences in phase and conjugate trajectories (as well as in terminal variables) are strong.*

References

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