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ORDINARY DIFFERENTIAL EQUATIONS

Nonclassical Relaxation Oscillations in a Mathematical Predator–Prey Model

S. D. Glyzin[1*], A. Yu. Kolesov[1], and N. Kh. Rozov[2***]**

[1] *Demidov Yaroslavl State University, Yaroslavl, 150003 Russia*

[2] *Lomonosov Moscow State University, Moscow, 119991 Russia*

*e-mail: *glyzin@uniyar.ac.ru, glyzin.s@gmail.com, **andkolesov@mail.ru, kolesov@uniyar.ac.ru, ***fpo.mgu@mail.ru, nicolas_rozov@mail.ru*

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Abstract— We consider the well-known Bazykin–Svirezhev model describing the predator–prey interaction. This model is a system of two nonlinear ordinary differential equations with a small parameter multiplying one of the derivatives. The existence and stability of a so-called relaxation cycle in such a system are studied. A peculiar feature of such a cycle is that as the small parameter tends to zero, its fast component changes in a δ-like manner, while the slow component tends to some discontinuous periodic function.

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1. STATEMENT OF THE PROBLEM AND RESULTS

The currently available general theory of relaxation oscillations in multidimensional systems of ordinary differential equations can somewhat tentatively be divided into classical and nonclassical theories. The classical theory goes back to the paper [1] by Mishchenko and Pontryagin, who postulated its main notions. The subsequent development of this theory is reflected in the monograph [2], and it acquired a fairly complete form in [3].

The simplest example of a system in which one can observe classical relaxation oscillations is the system of van der Pol equations, which has the form

$$\varepsilon\dot{x} = y - \frac{1}{3}x^3 + x, \quad \dot{y} = -x, \quad 0 < \varepsilon \ll 1. \tag{1.1}$$

As was shown in the monograph [2, Ch. 3], for all sufficiently small values of the parameter ε system (1.1) admits a stable relaxation cycle $(x(t,\varepsilon), y(t,\varepsilon))$, $x(0,\varepsilon) \equiv -3/2$, with period $T(\varepsilon)$ such that

$$\lim_{\varepsilon\to 0} T(\varepsilon) = T_0, \quad \lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t), \quad \lim_{\varepsilon\to 0} y(t,\varepsilon) = y_0(t). \tag{1.2}$$

Here $T_0 > 0$ is finite, and the functions $x_0(t)$ and $y_0(t)$ are T_0-periodic. Further, $y_0(t)$ is continuous, and $x_0(t)$ has two discontinuities of the first kind on any time interval of length equal to the period.

The first relation in (1.2) is preserved for a nonclassical relaxation cycle but, unlike the classical relaxation oscillations, now the fast component changes in a δ-like manner as $\varepsilon \to 0$, while the slow one tends to some discontinuous T_0-periodic function. In the case of relaxation systems on the plane, the theory of oscillations was set forth in the paper [4]. The results in this paper were included in extended form in the monograph [3]. One should also mention the paper [5] dealing with nonclassical relaxation oscillations in a mathematical model of the Belousov reaction.

The present paper studies the existence and stability of a nonclassical relaxation cycle in the well-known Bazykin–Svirezhev model describing the interaction of a highly prolific predator with its prey. This mathematical model has the form (see [6])

$$\begin{aligned} \dot{N}_1 &= r_1\left[1 - \frac{N_1}{K}\right]N_1^2 - aN_1N_2, \\ \dot{N}_2 &= r_2\left[\frac{N_1}{K} - b\right]N_2, \end{aligned} \tag{1.3}$$

where $N_1(t)$ and $N_2(t)$ are the predator and prey population sizes, respectively, and r_1, r_2, a, b, and K are positive parameters.

For convenience in the subsequent analysis, we make the changes of variables

$$Kr_1 t \to t, \quad N_1 = Kx, \quad N_2 = \frac{Kr_1}{a}y,$$

which reduce system (1.3) to the form

$$\begin{aligned}\dot{x} &= (1-x)x^2 - xy,\\ \dot{y} &= r(x-\delta)y,\end{aligned}$$

where $r = r_2/(Kr_1)$ and $\delta = b$. Further, assume that $r \gg 1$ (i.e., the predator is highly prolific). As a result, we arrive at the singularly perturbed system

$$\begin{aligned}\dot{x} &= (1-x)x^2 - xy,\\ \varepsilon\dot{y} &= (x-\delta)y,\end{aligned} \tag{1.4}$$

where $\varepsilon = 1/r \ll 1$. The resulting system, which can be viewed as a counterpart of the model system (1.1) in the theory of nonclassical relaxation oscillations, will be studied under the additional assumption

$$0 < \delta < 1/2. \tag{1.5}$$

It will become clear from the subsequent analysis that this restriction guarantees the existence of at least one stable nonclassical relaxation cycle in the system.

To find the cycles of system (1.4), we arbitrarily fix an $x_0 \in (\delta, 1]$ (this half-open interval exists by virtue of assumption (1.5)) and denote the trajectory of the system with the initial conditions $x(0, x_0, \varepsilon) = x_0$, $y(0, x_0, \varepsilon) = 1$ by

$$\Gamma(\varepsilon) = \{(x, y) : x = x(t, x_0, \varepsilon), \quad y = y(t, x_0, \varepsilon), \quad t \geq 0\}. \tag{1.6}$$

Further, consider the second positive root $t = T(x_0, \varepsilon)$ of the equation $y(t, x_0, \varepsilon) = 1$ (provided it exists) and define the Poincaré map by the formula

$$x_0 \mapsto \Pi(x_0, \varepsilon) \stackrel{\text{def}}{=} x(t, x_0, \varepsilon)|_{t=T(x_0,\varepsilon)}. \tag{1.7}$$

Our immediate goal is to establish how the operator (1.7) behaves asymptotically as $\varepsilon \to 0$.

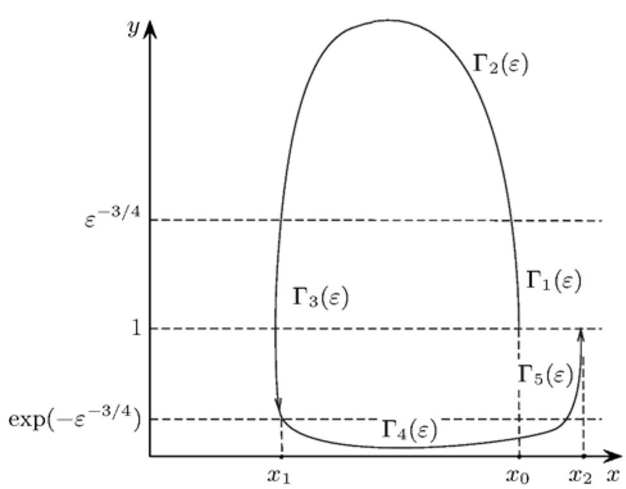

Fig. 1.1. Phase point trajectory.

Before we state rigorous assertions, we set forth some heuristic considerations. Note that in view of the choice of x_0, one has the inequality

$$(x_0 - \delta)y > 0 \quad \text{for all} \quad y > 0. \tag{1.8}$$

Hence, based on the form of system (1.4), we conclude that the phase point (x, y) first moves in an asymptotically small neighborhood of the ray $\{(x, y) : x = x_0,\ y \geq 1\}$, with the component $y(t, x_0, \varepsilon)$ increasing monotonously to reach the value $y = \varepsilon^{-3/4}$ in an asymptotically short time (of the order of $\varepsilon \ln(1/\varepsilon)$). We denote the corresponding part of the trajectory $\Gamma(\varepsilon)$ by $\Gamma_1(\varepsilon)$ (Fig. 1.1). Given inequality (1.8), it is appropriate to call this part the *takeoff segment*.

The next part of the trajectory lying in the half-plane $\{(x, y) : y \geq \varepsilon^{-3/4}\}$ will be denoted by $\Gamma_2(\varepsilon)$ and referred to as the *turning segment*. To find the trajectory behavior on this segment, we make the change of variables $u = \varepsilon y$ in (1.4) and adopt the new variable x for new time. As a result, after dropping asymptotically small terms, we obtain the following Cauchy problem for the function $u = u(x)$:

$$\frac{du}{dx} = -\frac{x - \delta}{x}, \quad u|_{x=x_0} = 0. \tag{1.9}$$

It can readily be seen that the solution of problem (1.9) is defined by the relation

$$u(x) = -(x - x_0) + \delta \ln \frac{x}{x_0}, \quad 0 < x \leq x_0. \tag{1.10}$$

Further, since we have

$$u(x_0) = 0, \quad u'(x) < 0 \quad \text{for} \quad \delta < x \leq x_0, \quad u'(x) > 0 \quad \text{for} \quad 0 < x < \delta,$$
$$u(x) \to -\infty \quad \text{as} \quad x \to +0,$$

we see that the equation $u(x) = 0$ admits a unique solution $x_1 = x_1(x_0)$ on the interval $(0, \delta)$, with $u(x) > 0$ for $x_1 < x < x_0$. It follows that when moving along the curve $\Gamma_2(\varepsilon)$, the phase point of system (1.4) first leaves the straight line $y = \varepsilon^{-3/4}$ and then returns to this line, i.e., makes a turn (see Fig. 1.1). Moreover, after passing to the variables (x, u), the concerned segment $\Gamma_2(\varepsilon) \subset \Gamma(\varepsilon)$ has the following curve as its limit as $\varepsilon \to 0$:

$$\Gamma_2^0 = \{(x, u) : u = u(x), \quad x_1 \leq x \leq x_0\}. \tag{1.11}$$

The form of the curve (1.11) is presented in Fig. 1.2.

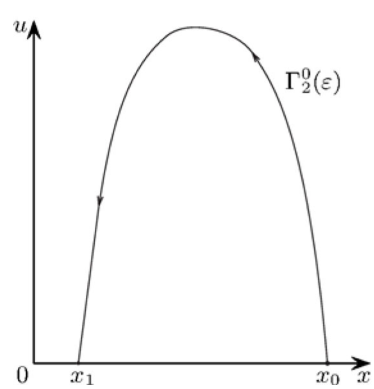

Fig. 1.2. Curve Γ_2^0.

It also needs to be mentioned that the time of motion along the segment $\Gamma_2(\varepsilon)$ is asymptotically small; strictly speaking, it is of the order of $\varepsilon \ln(1/\varepsilon)$. At the same time, each piece of the curve $\Gamma_2(\varepsilon)$ corresponding to values $x \in [a, b] \subset (x_1, x_0)$, where $a, b = \text{const} > 0$, is covered by the point (x, y) within a time period of the order of ε.

The next segment $\Gamma_3(\varepsilon)$ of the curve $\Gamma(\varepsilon)$, corresponding to values of the variable y in the interval $\exp(-\varepsilon^{-3/4}) \le y \le \varepsilon^{-3/4}$ is considered by analogy with how it was done for the segment $\Gamma_1(\varepsilon)$. Indeed, since

$$(x_1 - \delta)y < 0 \quad \text{for all} \quad y > 0, \tag{1.12}$$

we see that the curve $\Gamma_3(\varepsilon)$ is asymptotically close to the segment $\{(x, y) : x = x_1,\ \exp(-\varepsilon^{-3/4}) \le y \le \varepsilon^{-3/4}\}$ (see Fig. 1.1). However, since the variable y decreases here by virtue of (1.12), unlike $\Gamma_1(\varepsilon)$, it is appropriate to call this segment of the curve $\Gamma(\varepsilon)$ the *falling segment*. As for the "falling" time, it has the order of $\varepsilon^{1/4}$.

We will refer to the next segment $\Gamma_4(\varepsilon)$ of the curve $\Gamma(\varepsilon)$ lying in the half-plane $\{(x, y) : y \le \exp(-\varepsilon^{-3/4})\}$ (see Fig. 1.1) as the *slow motion segment*. To describe it, we switch in system (1.4) to the new variable $v = \varepsilon \ln y$ and adopt the variable x for new time. As a result, after dropping asymptotically small terms, we arrive at the following Cauchy problem for finding the function $v = v(x)$:

$$\frac{dv}{dx} = \frac{x - \delta}{(1 - x)x^2}, \quad v|_{x=x_1} = 0. \tag{1.13}$$

Solving this problem, we obtain

$$v(x) = (1 - \delta)\left(\ln \frac{x}{x_1} - \ln \frac{1 - x}{1 - x_1}\right) + \delta\left(\frac{1}{x} - \frac{1}{x_1}\right), \quad x_1 \le x < 1. \tag{1.14}$$

Note some of the properties of the solution (1.14) of problem (1.13). An easy check shows that $v'(x) < 0$ for $x_1 \le x < \delta$, $v'(x) > 0$ for $\delta < x < 1$, and $v(x) \to +\infty$ as $x \to 1 - 0$. Consequently, the equation $v(x) = 0$ has a unique solution $x_2 = x_2(x_1)$ on the interval $\delta < x < 1$ with $v(x) < 0$ for $x_1 < x < x_2$.

These properties of the function $v(x)$ allow one to claim that the graph of the curve $\Gamma_4(\varepsilon)$ indeed has the form depicted in Fig. 1.1. In addition, by analogy with the segment $\Gamma_2(\varepsilon)$, after passing to the variables (x, v), this curve (Fig. 1.3) tends as $\varepsilon \to 0$ to the limit

$$\Gamma_4^0 = \{(x, v) : v = v(x), \quad x_1 \le x \le x_2\}.$$

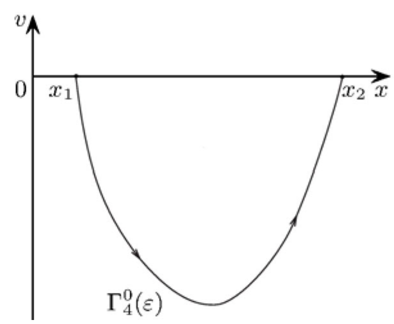

Fig. 1.3. Curve Γ_4^0.

The question about the time of motion along the curve $\Gamma_4(\varepsilon)$ deserves a separate note. Unlike all the preceding segments, here this time is asymptotically finite; i.e., it admits, as $\varepsilon \to 0$, a finite limit

$$T_0 = \ln \frac{x_2(1 - x_1)}{x_1(1 - x_2)} + \frac{1}{x_1} - \frac{1}{x_2}; \tag{1.15}$$

this explains why this segment has been called the slow motion segment.

At the final stage, consider the segment $\Gamma_5(\varepsilon) \subset \Gamma(\varepsilon)$, which will be called the *ascent segment* and which is located in the strip $\{(x, y) : \exp(-\varepsilon^{-3/4}) \le y \le 1\}$ (see Fig. 1.1). Note that since in this case one has the inequality obtained from (1.8) by replacing x_0 with x_2, it follows that the curve $\Gamma_5(\varepsilon)$ is asymptotically close to the segment $\{(x, y) : x = x_2,\ \exp(-\varepsilon^{-3/4}) \le y \le 1\}$, and in the course of motion along the $\Gamma_5(\varepsilon)$, the variable y increases. (That is why it is the ascent segment.) It is worth adding that the "ascent" time is asymptotically small (of the order of $\varepsilon^{1/4}$).

Rigorous justification of the above-listed facts pertaining to the asymptotic behavior of the trajectory (1.6) is based on the general theory of nonclassical relaxation oscillations developed in [3–5]. In particular, by repeating the proof of a similar result in the paper [5] almost word for word (with some simplifications), we arrive at the following assertion.

Theorem 1.1. *For each fixed* $\overline{\delta} \in (\delta, 1)$ *and for all sufficiently small* $\varepsilon > 0$, *the operator* (1.7) *is defined on the interval* $\overline{\delta} \le x_0 \le 1$ *and satisfies the limit relations*

$$\begin{gathered}\lim_{\varepsilon\to 0} \max_{\overline{\delta}\le x_0\le 1} |\Pi(x_0,\varepsilon) - \Pi(x_0)| = 0,\\ \lim_{\varepsilon\to 0} \max_{\overline{\delta}\le x_0\le 1} |\Pi'(x_0,\varepsilon) - \Pi'(x_0)| = 0.\end{gathered} \tag{1.16}$$

Here the prime denotes the derivative with respect to the variable x_0, *and the operator* $\Pi(x_0)$ *is defined by the relations*

$$\Pi = \Pi_2 \circ \Pi_1, \quad \Pi_1 : x_0 \mapsto x_1 = x_1(x_0), \quad \Pi_2 : x_1 \mapsto x_2 = x_2(x_1), \tag{1.17}$$

where $x_1 = x_1(x_0)$ *and* $x_2 = x_2(x_1)$ *are the above-introduced roots of the equations* $u(x) = 0$ *and* $v(x) = 0$, *respectively.*

The above theorem reduces the question about existence and stability of cycles in system (1.4) to a similar question for fixed points of the limit operator (1.17) that lie in the interval $\delta < x_0 < 1$. Namely, the following assertion holds.

Theorem 1.2. *Suppose that the mapping* (1.17) *admits a fixed point* $x_0 = x_0^* \in (\delta, 1)$ *such that* $|\Pi'(x_0^*)| < 1$. *Then in the original system* (1.4) *for all sufficiently small* $\varepsilon > 0$ *this fixed point is associated with the exponentially orbitally stable nonclassical relaxation cycle*

$$(x_*(t,\varepsilon), y_*(t,\varepsilon)), \quad y_*(0,\varepsilon) \equiv 1 \tag{1.18}$$

of period $T_*(\varepsilon)$.

We point out that Theorem 1.2 does not require separate justification, since it is a corollary of Theorem 1.1. Indeed, it follows from the limit relations (1.16) that the Poincaré map (1.7) for all sufficiently small $\varepsilon > 0$ admits an exponentially stable fixed point $x_0 = x_0^*(\varepsilon)$ asymptotically close to x_0^*. In system (1.4), the indicated fixed point is associated with the stable cycle (1.18) of period $T_*(\varepsilon)$, where

$$\begin{gathered}x_*(t,\varepsilon) = x(t,x_0,\varepsilon)|_{x_0=x_0^*(\varepsilon)},\\ y_*(t,\varepsilon) = y(t,x_0,\varepsilon)|_{x_0=x_0^*(\varepsilon)},\\ T_*(\varepsilon) = T(x_0,\varepsilon)|_{x_0=x_0^*(\varepsilon)}.\end{gathered} \tag{1.19}$$

It only remains to verify that the cycle is nonclassical.

Considering relations (1.19) and taking into account the above-described asymptotic properties of the trajectory (1.6), we conclude that the indicated cycle has the following properties. First, one has the limit relation

$$\lim_{\varepsilon\to 0} T_*(\varepsilon) = T_* \tag{1.20}$$

(where T_* is the quantity in (1.15) with $x_0 = x_0^*$ and $x_1 = x_1^* \equiv x_1(x_0^*)$).

Second, the component $y_*(t,\varepsilon)$ behaves in a δ-like manner as $\varepsilon \to 0$. Namely, on the interval $0 \le t \le t_*(\varepsilon)$, where $t_*(\varepsilon) = O(\varepsilon \ln(1/\varepsilon))$ is the first positive root of the equation $y_*(t,\varepsilon) = 1$, an asymptotically high burst (of the order of ε^{-1}) occurs with

$$\lim_{\varepsilon\to 0} \int_0^{t_*(\varepsilon)} y_*(t,\varepsilon)\,dt = \ln\frac{x_0^*}{x_1^*} > 0. \tag{1.21}$$

For the rest of the time, the function $y_*(t,\varepsilon)$ is exponentially small. To be precise, for any $\mu_1, \mu_2 \in (0, T_*/2)$ on the interval $\mu_1 \le t \le T_*(\varepsilon) - \mu_2$ one has the estimate

$$y_*(t,\varepsilon) \le M_1 \exp(-M_2/\varepsilon), \quad M_1, M_2 = \text{const} > 0. \tag{1.22}$$

Third, as $\varepsilon \to 0$, the component $x_*(t,\varepsilon)$ converges uniformly in t on an interval of the form $[\mu_1, T_*(\varepsilon) - \mu_2]$, where, as above, $\mu_1, \mu_2 = \text{const} \in (0, T_*/2)$, to a function $x_*(t)$ that is a solution of the Cauchy problem

$$\dot{x} = (1-x)x^2, \quad x|_{t=0} = x_1^*.$$

We point out that when periodically continued to the entire time axis with period T_*, the function $x_*(t)$ turns out to be discontinuous at the points $t = kT_*$, $k \in \mathbb{Z}$.

Combining properties (1.20)–(1.22) and considering the nature of the dependence of the component $x_*(t,\varepsilon)$ on ε, we conclude that the stable relaxation cycle (1.18) of system (1.4) delivered by Theorem 1.2 is nonclassical indeed. A visual idea of this cycle is given by Fig. 1.4, which provides the graphs of the components of this cycle for parameter values $\varepsilon = 0.01$ and $\delta = 0.44$ (the solid line indicates the graph of $y_*(t+c,\varepsilon)$ and the dashed line, the graph of $x_*(t+c,\varepsilon)$, where $c \in \mathbb{R}$ is some phase shift).

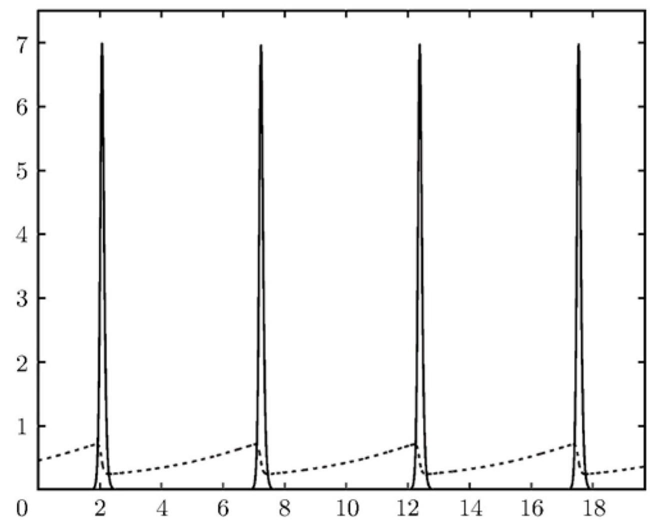

Fig. 1.4. Components of the stable relaxation cycle.

2. ANALYSIS OF THE LIMIT MAPPING

In this section, we verify that the applicability domain of Theorem 1.2 is a fortiori nonempty. To be precise, we establish the following assertion.

Theorem 2.1. *Under condition* (1.5), *the limit mapping* (1.17) *admits at least one exponentially stable fixed point on the interval* $\delta < x_0 < 1$.

We start the **proof** of this theorem by a remark on some general properties of the mapping (1.17). It follows from definition (1.17) and the way in which the functions $x_1(x_0)$ and $x_2(x_1)$ have been defined that, first,

$$\Pi'(x_0) = \frac{1-x_2}{1-x_1}\frac{x_0-\delta}{x_2-\delta}\frac{x_2^2}{x_0x_1}\bigg|_{\substack{x_1=x_1(x_0)\\ x_2=x_2(x_1)}} > 0 \quad \text{for all} \quad x_0 \in (\delta, 1]; \tag{2.1}$$

second,

$$\lim_{x_0 \to \delta+0} \Pi(x_0) = \delta, \quad \Pi(1) < 1. \tag{2.2}$$

Relations (2.1) and (2.2) show that when extended by continuity to the value $x_0 = \delta$, the mapping (1.17) takes the segment $\delta \le x_0 \le 1$ into itself and has the fixed point $x_0 = \delta$. Let us verify that this fixed point is unstable.

First, consider the function

$$x_1(\nu) = x_1(x_0)|_{x_0=\delta+\nu}, \quad 0 < \nu \ll 1. \tag{2.3}$$

Substituting the expression $x_0 = \delta + \nu$ into relation (1.10) and analyzing the resulting equation $u(x) = 0$, we conclude that the desired function (2.3) admits the asymptotics

$$x_1(\nu) = \delta - \nu + \frac{2}{3\delta}\nu^2 + O(\nu^3) \quad \text{as} \quad \nu \to +0. \tag{2.4}$$

Now let us substitute the representation (2.4) into (1.14) and carry out an asymptotic calculation of the function $x_2(\nu) = x_2(x_1)|_{x_1=x_1(\nu)}$, which, as we remember, is a root of the corresponding equation $v(x) = 0$. As a result, we obtain

$$\Pi(\delta+\nu) = x_2(\nu) = \delta + \nu + \frac{2}{3\delta}\frac{1-2\delta}{1-\delta}\nu^2 + O(\nu^3) \quad \text{as} \quad \nu \to +0. \tag{2.5}$$

Relation (2.5) and condition (1.5) guarantee that the inequality $\Pi(\delta+\nu) > \delta+\nu$ for $0 < \nu \ll 1$ holds. This, in turn, implies that the fixed point $x_0 = \delta$ of the mapping (1.17) is repelling. In the original system (1.4), this point is associated with the equilibrium $(x, y) = (\delta, \delta(1-\delta))$, which is also unstable under condition (1.5).

Combining the facts listed above, we conclude that the mapping (1.17) transforms an interval of the form $\overline{\delta} \le x_0 \le 1$ strictly into itself, where the quantity $\overline{\delta}$ belongs to the interval $(\delta, 1)$ and is sufficiently close to δ. Since the mapping $\Pi(x_0)$ is analytic, it follows that, first, the number of fixed points of this mapping on the indicated interval is finite and the fixed point set is a fortiori nonempty; second, among these points there exists at least one fixed point $x_0 = x_0^*$ such that the difference $\Pi(x_0) - x_0$ changes its sign from "+" to "−" when passing through this point. Since the mapping $\Pi(x_0)$ is monotone (see (2.1)), it follows that this fixed point is asymptotically stable, and it is exponentially stable under the condition $\Pi'(x_0^*) \neq 1$.

The above preliminary analysis implies that to justify Theorem 2.1, it suffices to check whether the inequality

$$\Pi'(x_0) \neq 1 \tag{2.6}$$

holds for each fixed point $x_0 \in (\delta, 1)$ of the mapping (1.17).

Assume the contrary: there exists a value $x_0 \in (\delta, 1)$ such that

$$\Pi(x_0) = x_0, \quad \Pi'(x_0) = 1. \tag{2.7}$$

Since the point x_0 is fixed, we have $x_2 = x_0$. Taking into account this relation in the formula for $\Pi'(x_0)$ (see (2.1)), we conclude that the second relation in (2.7) is equivalent to the relation

$$x_0(1-x_0) = x_1(1-x_1).$$

Hence we necessarily have

$$x_1 = 1 - x_0. \tag{2.8}$$

Substituting the expression (2.8) into the equation $u(x_1) = 0$, we arrive at the relation

$$\delta = \frac{2x_0 - 1}{\ln(x_0/(1-x_0))}, \tag{2.9}$$

which allows one to localize possible values of the variable x_0.

To this end, we introduce the function

$$\omega(x_0) = \frac{2x_0 - 1}{\ln(x_0/(1-x_0))} \tag{2.10}$$

and note that

$$
\begin{gathered}
\lim_{x_0\to+0}\omega(x_0)=0,\\
\lim_{x_0\to1/2}\omega(x_0)=1/2,\\
\lim_{x_0\to1-0}\omega(x_0)=0,\\
\omega'(x_0)>0 \quad \text{for each} \quad x_0\in(0,1/2),\\
\omega'(x_0)<0 \quad \text{for each} \quad x_0\in(1/2,1).
\end{gathered} \tag{2.11}
$$

Indeed, the first three properties in (2.11) automatically follow from the explicit form (2.10) of the function $\omega(x_0)$. At the same time, the inequalities can easily be reduced to the conditions

$$
\begin{gathered}
\varphi(x_0)>0 \quad \text{for each} \quad x_0\in(0,1/2),\\
\varphi(x_0)<0 \quad \text{for each} \quad x_0\in(1/2,1),
\end{gathered} \tag{2.12}
$$

where

$$
\varphi(x_0)=2x_0(1-x_0)\ln\frac{x_0}{1-x_0}-2x_0+1. \tag{2.13}
$$

In turn, it follows from (2.13) that

$$
\varphi(1/2)=0,\quad \varphi'(x_0)=2(1-2x_0)\ln\frac{x_0}{1-x_0}<0 \quad \text{for each} \quad x_0\in(0,1),\quad x_0\neq1/2;
$$

the latter in a self-obvious manner implies the desired inequalities (2.12).

Taking into account condition (1.5) and property (2.11), we conclude that the variable x_0 in (2.9) may at best belong to the union of the intervals $(0,1/2)$ and $(1/2,1)$. However, it may not belong to the interval $(0,1/2)$, because on this interval one has the inequality

$$
\ln\frac{x_0}{1-x_0}>2-\frac{1}{x_0},
$$

contradicting the requirement $x_0>\delta=\omega(x_0)$.

Now consider the equation $v(x)=0$ for finding $x=x_2$. Substituting the expressions $x=x_0$ and $x_1=1-x_0$ into this equation, we arrive at the relation

$$
\delta=2\left[\frac{2x_0-1}{x_0(1-x_0)}+2\ln\frac{x_0}{1-x_0}\right]^{-1}\ln\frac{x_0}{1-x_0}, \tag{2.14}
$$

which is similar to (2.9). Equating the right-hand sides of relations (2.9) and (2.14) with each other, we verify that x_0 is necessarily a root of the equation

$$
\Phi(x_0)\stackrel{\text{def}}{=}2\ln^2\frac{x_0}{1-x_0}-2(2x_0-1)\ln\frac{x_0}{1-x_0}-\frac{(2x_0-1)^2}{x_0(1-x_0)}=0. \tag{2.15}
$$

However, as will be shown below, this equation does not have solutions on the interval $1/2<x_0<1$.

In the analysis to follow, we need the obvious relations

$$
\Phi'(x_0)=4\left(\frac{1}{x_0}+\frac{1}{1-x_0}-1\right)\ln\frac{x_0}{1-x_0}-6\frac{2x_0-1}{x_0(1-x_0)}-\frac{(2x_0-1)^3}{x_0^2(1-x_0)^2}, \tag{2.16}
$$

$$
x_0^2(1-x_0)^2\Phi''(x_0)=2(2x_0-1)\left[2\ln\frac{x_0}{1-x_0}-3(2x_0-1)-(2x_0-1)\frac{x_0^3+(1-x_0)^3}{x_0(1-x_0)}\right], \tag{2.17}
$$

which will be used to establish the properties

$$
\Phi'(x_0)<0,\quad \Phi''(x_0)<0 \quad \text{for each} \quad x_0\in(1/2,1). \tag{2.18}
$$

First, consider the second inequality in (2.18) and note that, by virtue of (2.17), it is equivalent to the estimate

$$\frac{2x_0(1-x_0)}{3x_0(1-x_0)+x_0^3+(1-x_0)^3}\ln\frac{x_0}{1-x_0} < 2x_0-1 \quad \text{for each} \quad x_0\in(1/2,1). \tag{2.19}$$

Let us make the change of variables $x_0/(1-x_0)=z_0\in(1,+\infty)$ in (2.19), which transforms the desired estimate to the form

$$\psi(z_0) < z_0-1 \quad \text{for each} \quad z_0\in(1,+\infty), \tag{2.20}$$

where

$$\psi(z_0)=\frac{2z_0}{1+z_0}\ln z_0. \tag{2.21}$$

A straightforward verification shows that the function (2.21) has the properties

$$\psi(1)=0,\quad \psi'(1)=1,\quad \psi''(z_0)<0 \quad \text{for each} \quad z_0>1,$$

and hence for $z_0>1$ its graph $x=\psi(z_0)$ lies below its tangent straight line $x=z_0-1$; i.e., the inequality (2.20) is satisfied.

Now the first inequality in (2.18) in an obvious manner follows from the above-derived estimate $\Phi''(x_0)<0$, where $x_0\in(1/2,1)$, and from the easy-to-check relation $\Phi'(1/2)=0$ (see (2.16)). Further, by virtue of the first inequality in (2.18) and the fact that $\Phi(1/2)=0$ (see (2.15)), we conclude that $\Phi(x_0)<0$ for all $x_0\in(1/2,1)$. Thus, Eq. (2.15) does not have solutions on the interval $(1/2,1)$, and hence assumptions (2.7) are false. The proof of the theorem is complete.

Apparently, the assertion about the uniqueness of a stable fixed point of the mapping (1.17) holds true. According to the constructions above, to prove this assertion it suffices to establish that each fixed point x_0 of this mapping on the interval $(\delta,1)$ satisfies the condition $\Pi'(x_0)<1$ instead of condition (2.6). Another method for proving the uniqueness is as follows.

When seeking a solution $x=x_1$ of the equation $u(x)=0$, we introduce the new variable $z=x_1/x_0$, which, by virtue of the inequality $x_1<x_0$, belongs to the interval $(0,1)$. As follows from the equation $u(x)=0$, for x_0 one has the relation

$$x_0=\frac{\delta\ln z}{z-1}. \tag{2.22}$$

Note also that in the case of (2.22), the estimate $x_0>\delta$ is satisfied automatically, but one needs to require specifically that the inequality $x_0<1$ be satisfied. It can readily be seen that this inequality is equivalent to the condition

$$z-1-\delta\ln z<0. \tag{2.23}$$

Now consider the equation $v(x)=0$ for finding $x=x_2$. Since for each fixed point of the mapping (1.17) one has the relation $x_2=x_0$, we substitute the expressions

$$x_2=x_0,\quad x_1=zx_0,\quad x_0=\frac{\delta\ln z}{z-1}$$

into the relation $v(x_2)=0$. As a result, to find the as yet unknown quantity z, we arrive at the equation

$$\Psi(z)\stackrel{\text{def}}{=}-(1-\delta)\left(\ln z+\ln\frac{z-1-\delta\ln z}{z-1-\delta z\ln z}\right)+\frac{(z-1)^2}{z\ln z}=0. \tag{2.24}$$

It follows from the above constructions that there is a one-to-one correspondence between the roots of Eq. (2.24) satisfying inequality (2.23) and the fixed points of the mapping (1.17) belonging to the interval $(\delta,1)$. Let us write condition (2.23) in the equivalent form

$$z_0(\delta)<z<1, \tag{2.25}$$

where $z_0(\delta)\in(0,1)$ is the unique root of the equation $z-1-\delta\ln z=0$.

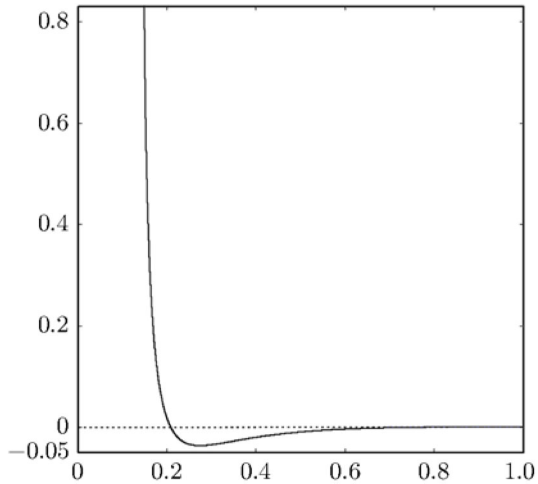

Fig. 2.1. Graph of the function $\Psi(z)$.

An easy verification shows that the function $\Psi(z)$ defined in (2.24) has the properties

$$\lim_{z\to z_0(\delta)+0}\Psi(z)=+\infty,$$
$$\Psi(1-\nu)=\frac{\nu^3}{12}\frac{2\delta-1}{(1-\delta)^2}+O(\nu^4),\quad 0<\nu\ll 1.$$

Thus, under condition (1.5), Eq. (2.24) necessarily has at least one root on the interval (2.25). As is proved by numerical analysis, this root is unique (see Fig. 2.1, which shows the graph of the function $\Psi(z)$ on the interval (2.25) for $\beta=0.44$). However, we have not been able to justify this assertion rigorously.

3. BILOCAL MODEL

A natural generalization of the model (1.3) is the so-called bilocal model [6], which is a system of the form

$$\begin{aligned}\dot N_1&=r\left(1-\frac{N_1}{K}\right)N_1^2-aN_1N_3+D(N_2-N_1),\\ \dot N_2&=r\left(1-\frac{N_2}{K}\right)N_2^2-aN_2N_3+D(N_1-N_2),\\ \dot N_3&=\lambda\left(\alpha\frac{N_1}{K}+(1-\alpha)\frac{N_2}{K}-b\right)N_3.\end{aligned}\qquad(3.1)$$

Here N_1 and N_2 are the prey population sizes, N_3 is the predator population size, all the parameters r, λ, a, b, D, and α are positive, and, in addition, $\alpha\in(0,1)$. The quantities α and $1-\alpha$ are the fractions of prey N_1 and N_2, respectively, in the ration of the predator N_3.

It can readily be seen that after some normalization system (3.1) can be reduced to a form similar to (1.4), namely,

$$\begin{aligned}\dot\xi_1&=(1-\xi_1)\xi_1^2-\xi_1y+d(\xi_2-\xi_1),\\ \dot\xi_2&=(1-\xi_2)\xi_2^2-\xi_2y+d(\xi_1-\xi_2),\\ \varepsilon\dot y&=(\alpha\xi_1+(1-\alpha)\xi_2-\delta)y,\end{aligned}\qquad(3.2)$$

where $0<\varepsilon\ll 1$, $d,\delta=\text{const}>0$, and $\alpha=\text{const}\in(0,1)$. It turns out that analogs of Theorems 1.1 and 1.2 hold for the resulting system (3.2). Below we give the corresponding results in a concise form.

Fix an arbitrarily point $\xi_{(0)} = (\xi_{1,0}, \xi_{2,0}) \in \mathbb{R}^2$ whose coordinates satisfy the conditions

$$\xi_{1,0}, \xi_{2,0} > 0, \quad x_0 \stackrel{\text{def}}{=} \alpha\xi_{1,0} + (1-\alpha)\xi_{2,0} > \delta. \tag{3.3}$$

Further, let

$$\Gamma(\varepsilon) = \{(\xi_1, \xi_2, y) : \xi_j = \xi_j(t, \xi_{(0)}, \varepsilon), \quad j = 1, 2, \quad y = y(t, \xi_{(0)}, \varepsilon), \quad t \geq 0\} \tag{3.4}$$

be the trajectory of system (3.2) with the initial conditions

$$\xi_j(0, \xi_{(0)}, \varepsilon) = \xi_{j,0}, \quad j = 1, 2, \quad y(0, \xi_{(0)}, \varepsilon) = 1.$$

We introduce the second positive root $t = T(\xi_{(0)}, \varepsilon)$ of the equation $y(t, \xi_{(0)}, \varepsilon) = 1$ (if it exists) and define a Poincaré map similar to (1.7) by

$$\xi_{(0)} \mapsto \Pi(\xi_{(0)}, \varepsilon) \stackrel{\text{def}}{=} (\xi_1(t, \xi_{(0)}, \varepsilon), \xi_2(t, \xi_{(0)}, \varepsilon))|_{t=T(\xi_{(0)},\varepsilon)}. \tag{3.5}$$

Our immediate goal is to describe the limit mapping $\Pi(\xi_{(0)})$ to which this operator converges (in the C^1-metric) as $\varepsilon \to 0$.

Let us divide the trajectory $\Gamma(\varepsilon)$ into the same segments $\Gamma_j(\varepsilon)$, $j = 1, \ldots, 5$, as in the case of the trajectory (1.6). Note that the quantity x_0 in (3.3) satisfies inequality (1.8). It follows that on the takeoff segment $\Gamma_1(\varepsilon)$ the phase point $(\xi_1,\ \xi_2\ y)$ moves in an asymptotically small neighborhood of the ray $\{(\xi_1, \xi_2, y) : \xi_1 = \xi_{1,0},\ \xi_2 = \xi_{2,0},\ y \geq 1\}$, and in a time of the order of $\varepsilon \ln(1/\varepsilon)$ the component $y(t, \xi_{(0)}, \varepsilon)$ monotonously increases to reach the value $\varepsilon^{-3/4}$.

On the segment $\Gamma_2(\varepsilon)$ of the trajectory (3.4) lying in the half-space $\{(\xi_1, \xi_2, y) : y \geq \varepsilon^{-3/4}\}$, in system (3.2) we make the change of variables $y = u/\varepsilon$ and adopt the variable $x = \alpha\xi_1 + (1-\alpha)\xi_2$ for the new time. As a result, after discarding asymptotically small terms, we obtain a Cauchy problem similar to (1.9) for the components u, ξ_j, $j = 1, 2$; namely,

$$\frac{du}{dx} = -\frac{x-\delta}{x}, \quad u|_{x=x_0} = 0, \quad \frac{d\xi_j}{dx} = \frac{\xi_j}{x}, \quad \xi_j|_{x=x_0} = \xi_{j,0}, \quad j = 1, 2,$$

whose solution is determined by the relations

$$u = u(x), \quad \xi_j = \xi_{j,0}\frac{x}{x_0}, \quad j = 1, 2, \tag{3.6}$$

where $u(x)$ is the function (1.10).

It should also be added that, first, formulas (3.6) hold only on the interval $x_1 \leq x \leq x_0$, where, as we remember, x_1 is the root of the equation $u(x) = 0$ in the interval $(0, \delta)$; second, after switching to the variables u, ξ_j, $j = 1, 2$, the curve $\Gamma_2(\varepsilon) \subset \Gamma(\varepsilon)$ tends, as $\varepsilon \to 0$, to the limit curve

$$\Gamma_2^0 = \{(\xi_1, \xi_2, u) : u = u(x), \quad \xi_j = \xi_{j,0}x/x_0, \quad j = 1, 2, \quad x_1 \leq x \leq x_0\};$$

and third, the time of motion along $\Gamma_2(\varepsilon)$ is asymptotically small (more precisely, it is of the order of $\varepsilon \ln(1/\varepsilon)$).

The next segment $\Gamma_3(\varepsilon)$ resides in the strip $\{(\xi_1, \xi_2, y) : \exp(-\varepsilon^{-3/4}) \leq y \leq \varepsilon^{-3/4}\}$. Since the vector $\xi_{(1)} = (\xi_{1,1}, \xi_{2,1})$, where

$$\xi_{j,1} = \xi_{j,0}\frac{x_1}{x_0}, \quad j = 1, 2, \tag{3.7}$$

satisfies the inequality $\alpha\xi_{1,1} + (1-\alpha)\xi_{2,1} = x_1 < \delta$ on the segment indicated, it follows that the phase point (ξ_1, ξ_2, y) moves in an asymptotically small neighborhood of the segment

$$\left\{(\xi_1, \xi_2, y) : \xi_j = \xi_{j,1}, \quad j = 1, 2, \quad \exp\left(-\varepsilon^{-3/4}\right) \leq y \leq \varepsilon^{-3/4}\right\},$$

with the component y decreasing. At the same time, the time of motion along $\Gamma_3(\varepsilon)$ is of the order of $\varepsilon^{1/4}$.

It is appropriate to refer to the segment $\Gamma_4(\varepsilon)$ of the trajectory (3.4) lying in the half-space $\{(\xi_1,\xi_2,y) : y \le \exp(-\varepsilon^{-3/4})\}$ as the slow motion segment by analogy with the case of the curve (1.6). When investigating this segment, in system (3.2) we switch to the variable $v = \varepsilon \ln y$. As a result, after dropping asymptotically small terms, we obtain the system

$$\dot{\xi}_1 = (1-\xi_1)\xi_1^2 + d(\xi_2 - \xi_1), \quad \dot{\xi}_2 = (1-\xi_2)\xi_2^2 + d(\xi_1 - \xi_2), \tag{3.8}$$

$$\dot{v} = \alpha\xi_1 + (1-\alpha)\xi_2 - \delta. \tag{3.9}$$

As follows from the consideration of the previous segment, this system should be equipped with the initial conditions

$$\xi_j|_{t=0} = \xi_{j,1}, \quad j = 1,2, \quad v|_{t=0} = 0, \tag{3.10}$$

where the $\xi_{j,1}$ are the components in (3.7).

Let $(\xi_1(t,\xi_{(1)}), \xi_2(t,\xi_{(1)}))$, $t \ge 0$, be the solution of subsystem (3.8) with the initial condition $(\xi_1,\xi_2)|_{t=0} = \xi_{(1)}$. Then it obviously follows from (3.9) and (3.10) that

$$v(t) = \int_0^t [\alpha\xi_1(\tau,\xi_{(1)}) + (1-\alpha)\xi_2(\tau,\xi_{(1)}) - \delta]\, d\tau. \tag{3.11}$$

When analyzing the function (3.11), we will assume the following condition to be satisfied:

$$0 < \delta < 1. \tag{3.12}$$

Then this function has the properties

$$\begin{gathered} v(0) = 0, \quad \dot{v}(0) = \alpha\xi_{1,1} + (1-\alpha)\xi_{2,1} - \delta = x_1 - \delta < 0, \quad \text{and} \\ v(t) \to +\infty \quad \text{as} \quad t \to +\infty. \end{gathered} \tag{3.13}$$

Indeed, the first two of these properties are self-obvious. To prove the third property in (3.13), note the following facts. First, the cone $\mathbb{R}^2_+ = \{(\xi_1,\xi_2) : \xi_1 > 0,\ \xi_2 > 0\}$ is invariant for the solutions of system (3.8). Second, system (3.8) is dissipative in this cone. (All of its trajectories eventually enter some set of the form $\mathbb{R}^2_+ \cap \{(\xi_1,\xi_2) : \xi_1^2 + \xi_2^2 \le r^2\}$, $r = \text{const} > 0$.) Third, system (3.8) in the cone $\mathbb{R}^2_+$ admits a unique exponentially stable equilibrium $\xi_1 = 1$, $\xi_2 = 1$ and cannot have cycles (because it has the invariant ray $\xi_1 = \xi_2$ passing through this equilibrium).

Based on the above information about system (3.8), we conclude that for each $\xi_{(1)} \in \mathbb{R}^2_+$ one has the relations

$$\lim_{t\to+\infty} \xi_1(t,\xi_{(1)}) = \lim_{t\to+\infty} \xi_2(t,\xi_{(1)}) = 1,$$

which imply that the integrand in (3.11) tends to the positive limit $1-\delta$ as $t \to +\infty$. Hence the third condition in (3.13) follows automatically. Further, in turn, it follows from relations (3.13) that there exists a $T_0 = T_0(\xi_{(1)}) > 0$ such that

$$v(t) < 0 \quad \text{for} \quad 0 < t < T_0, \quad v(T_0) = 0, \quad \dot{v}(T_0) \ge 0. \tag{3.14}$$

Throughout the following, we assume that the last inequality in (3.14) is strict; i.e.,

$$\dot{v}(T_0) > 0. \tag{3.15}$$

This inequality reflects some generality of the position and is a restriction for the choice of the initial vector $\xi_{(0)}$.

Concluding the consideration of the segment $\Gamma_4(\varepsilon)$, notice that after switching to the variables ξ_1, ξ_2, v, this segment tends as $\varepsilon \to 0$ to the limit

$$\Gamma_4^0 = \{(\xi_1,\xi_2,v) : \xi_1 = \xi_1(t,\xi_{(1)}), \quad \xi_2 = \xi_2(t,\xi_{(1)}), \quad v = v(t), \quad 0 \le t \le T_0\}.$$

As for the time of motion along the curve $\Gamma_4(\varepsilon)$, this time tends to T_0 as $\varepsilon \to 0$; i.e., it is asymptotically finite.

When considering the last segment $\Gamma_5(\varepsilon)$ of the trajectory (3.4), note that, by virtue of (3.15), the vector

$$\xi_{(2)} = (\xi_{1,2}, \xi_{2,2}) \stackrel{\text{def}}{=} (\xi_1(t, \xi_{(1)}), \xi_2(t, \xi_{(1)}))|_{t=T_0(\xi_{(1)})} \tag{3.16}$$

satisfies the inequality

$$(\alpha\xi_{1,2} + (1-\alpha)\xi_{2,2} - \delta)y > 0 \quad \text{for all} \quad y > 0. \tag{3.17}$$

In turn, it follows from inequality (3.17) that the variable y is monotone increasing on the segment $\Gamma_5(\varepsilon)$ from $\exp(-\varepsilon^{-3/4})$ to 1, while the components ξ_1 and ξ_2 are asymptotically close to the coordinates of the vector (3.16). The time in which this segment is covered is of the order of $\varepsilon^{1/4}$.

To make the above constructions rigorous, one needs to apply the general theory of nonclassical relaxation oscillations set forth in [3–5] once more. Based on the results in these publications, we conclude that the following assertion holds.

Theorem 3.1. *Assume that condition* (3.12) *is satisfied and one has arbitrarily fixed a compact set* $\Omega \subset \mathbb{R}^2_+$ *in which each vector* $\xi_{(0)} = (\xi_{1,0}, \xi_{2,0})$ *satisfies conditions* (3.3) *and* (3.15). *Then for all sufficiently small* $\varepsilon > 0$ *the operator* (3.5) *is defined on the set* Ω, *and one has the limit relations*

$$\begin{gathered} \lim_{\varepsilon\to 0} \max_{\xi_{(0)}\in\Omega} \|\Pi(\xi_{(0)}, \varepsilon) - \Pi(\xi_{(0)})\|_{\mathbb{R}^2} = 0, \\ \lim_{\varepsilon\to 0} \max_{\xi_{(0)}\in\Omega} \|\Pi'(\xi_{(0)}, \varepsilon) - \Pi'(\xi_{(0)})\|_{\mathbb{R}^2\to\mathbb{R}^2} = 0. \end{gathered} \tag{3.18}$$

Here the prime denotes the derivative with respect to the vector argument $\xi_{(0)}$, *and the limit operator* $\Pi(\xi_{(0)})$ *has the form*

$$\Pi = \Pi_2 \circ \Pi_1, \quad \Pi_1 : \xi_{(0)} \to \xi_{(1)}, \quad \Pi_2 : \xi_{(1)} \to \xi_{(2)}. \tag{3.19}$$

It follows from relations (3.18) that to each fixed point $\xi^*_{(0)} \in \Omega$, exponentially stable or dichotomic, of the limit operator (3.19) there corresponds a nonclassical relaxation cycle with the same stability properties in the original system (3.2).

Under conditions (1.5), system (3.2) obviously admits the so-called *homogeneous* cycle

$$\xi_1 = \xi_2 = x_*(t, \varepsilon), \quad y = y_*(t, \varepsilon), \tag{3.20}$$

where $x_*(t, \varepsilon)$ and $y_*(t, \varepsilon)$ are the components of the stable relaxation cycle (1.18) of system (1.4), whose existence is stated in Theorems 1.2 and 2.1. The question about the stability of this cycle is answered by Theorem 3.1, which implies the following result.

Theorem 3.2. *The homogeneous cycle* (3.20) *of system* (3.2) *is exponentially orbitally stable for all sufficiently small* $\varepsilon > 0$ *and for any fixed* $\alpha \in (0, 1)$, $d \geq 0$.

Proof. Let $x_0 = x^*_0$ be an exponentially stable fixed point of the mapping (1.17), whose existence is guaranteed by Theorem 2.1. Then it is obvious that the two-dimensional mapping (3.19) has the fixed point $\xi^*_{(0)} = (x^*_0, x^*_0)$, and conditions (3.3) and (3.15) are satisfied on the set

$$\Omega = \{(\xi_{1,0}, \xi_{2,0}) : (\xi_{1,0} - x^*_0)^2 + (\xi_{2,0} - x^*_0)^2 \leq r^2\}$$

provided that $r > 0$ is diminished appropriately. Note that the fixed point $\xi^*_{(0)}$ in the original system (3.2) is associated with the cycle (3.20). Based on this and Theorem 3.1, we conclude that the stability properties of this cycle are determined by the nature of the arrangement of the eigenvalues of the linear operator $V : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$V = \Pi'(\xi^*_{(0)}). \tag{3.21}$$

To carry out the linearization (3.21) of the mapping (3.19) at the fixed point $\xi^*_{(0)} = (x^*_0, x^*_0)$, we make some preliminary constructions. First, we substitute the expressions

$$\xi_{j,0} = x^*_0 + h_j, \quad j = 1, 2, \quad x_0 = x^*_0 + \Delta x_0, \quad x_1 = x^*_1 + \Delta x_1,$$

into formulas (3.7), where, we recall, $x_1^* \in (0,\delta)$ is the root of the equation $u(x) = 0$ for $x_0 = x_0^*$ (see (1.10)). Note that, according to the equality for x_0 in (3.3), the increments Δx_0 and Δx_1 in the linear approximation have the form

$$\begin{aligned} \Delta x_0 &= \alpha h_1 + (1-\alpha)h_2, \\ \Delta x_1 &= \frac{x_1^*(\delta - x_0^*)}{x_0^*(\delta - x_1^*)}\Delta x_0. \end{aligned} \tag{3.22}$$

Considering the above, we conclude that, up to terms that have order higher than first in h_1 and h_2, the following formulas hold:

$$\xi_{(1)} = \xi_{(1)}^* + \Delta\xi_{(1)}, \quad \xi_{(1)}^* = (x_1^*, x_1^*), \quad \Delta\xi_{(1)} = (\Delta\xi_{1,1}, \Delta\xi_{2,1}), \tag{3.23}$$

where

$$\Delta\xi_{j,1} = h_j\frac{x_1^*}{x_0^*} + \Delta x_1 - \frac{x_1^*}{x_0^*}\Delta x_0, \quad j = 1,2. \tag{3.24}$$

Now consider the vector $\xi_{(2)}$ and represent it in a form similar to (3.23) as

$$\xi_{(2)} = \xi_{(2)}^* + \Delta\xi_{(2)}, \quad \xi_{(2)}^* = (x_0^*, x_0^*), \quad \Delta\xi_{(2)} = (\Delta\xi_{1,2}, \Delta\xi_{2,2}), \tag{3.25}$$

where $\Delta\xi_{j,2}$, $j = 1,2$, are some functions linear in h_1 and h_2. To this end, we need the linear system

$$\begin{aligned} \dot g_1 &= (2x_*(t) - 3x_*^2(t))g_1 + d(g_2 - g_1), \\ \dot g_2 &= (2x_*(t) - 3x_*^2(t))g_2 + d(g_1 - g_2), \end{aligned} \tag{3.26}$$

in which, we remember, $x_*(t)$ is the zero approximation to the component $x_*(t,\varepsilon)$ of the cycle (1.18). More precisely, we will be interested in the solution of this system with the initial conditions

$$\begin{aligned} g_1|_{t=0} &= \Delta\xi_{1,1}, \\ g_2|_{t=0} &= \Delta\xi_{2,1}, \end{aligned} \tag{3.27}$$

where $\Delta\xi_{1,1}$ and $\Delta\xi_{2,1}$ are the increments (3.24).

A simple analysis shows that one has the following explicit formulas for the components of the solution of the Cauchy problem (3.26), (3.27):

$$\begin{aligned} g_1(t) &= \frac{\dot x_*(t)}{2(1-x_1^*)(x_1^*)^2}(\Delta\xi_{1,1} + \Delta\xi_{2,1} + (\Delta\xi_{1,1} - \Delta\xi_{2,1})\exp(-2\,dt)), \\ g_2(t) &= \frac{\dot x_*(t)}{2(1-x_1^*)(x_1^*)^2}(\Delta\xi_{1,1} + \Delta\xi_{2,1} - (\Delta\xi_{1,1} - \Delta\xi_{2,1})\exp(-2\,dt)). \end{aligned} \tag{3.28}$$

Based on this and (3.16) and considering the already established relations (3.22)–(3.24), we obtain the desired relations (3.25) in which

$$\begin{aligned} \Delta\xi_{j,2} &= g_j(T_*) + (1-x_0^*)(x_0^*)^2\Delta T, \quad j = 1,2, \\ \Delta T &= \frac{1}{\delta - x_0^*}\int_0^{T_*}(\alpha g_1(t) + (1-\alpha)g_2(t))\,dt \end{aligned} \tag{3.29}$$

and T_* is the quantity in (1.20). Finally, combining relations (3.25) and (3.29), we make sure that the linear operator (3.21) acts by the rule

$$V : (h_1, h_2) \mapsto (\Delta\xi_{1,2}, \Delta\xi_{2,2}). \tag{3.30}$$

Summarizing, note that the explicit formulas (3.22), (3.24), and (3.28)–(3.30) derived above make it possible to find the spectrum $\{\lambda_1,\ \lambda_2\}$ of the operator V. An appropriate calculation leads to the relations

$$\lambda_1 = \frac{x_0^*(1-x_0^*)}{x_1^*(1-x_1^*)},$$
$$\lambda_2 = \frac{x_0^*(1-x_0^*)}{x_1^*(1-x_1^*)}\exp(-2dT_*).$$

By virtue of the exponential stability of the fixed point x_0^* of the operator (1.17), we have

$$\frac{x_0^*(1-x_0^*)}{x_1^*(1-x_1^*)} \in (0,1),$$

and hence $\lambda_1, \lambda_2 \in (0,1)$. The proof of the theorem is complete.

It is of interest to note that under certain conditions stable relaxation cycles other than the homogeneous cycle (3.20) may exist in system (3.2). To elucidate what the matter is here, we first assume that

$$d = 0, \quad 0 < \delta < (1-\alpha)/2. \tag{3.31}$$

In this case, system (3.2) admits a stable cycle of the form

$$(\xi_1, \xi_2, y) = (0, \xi_*(t,\varepsilon), y_*(t,\varepsilon)), \tag{3.32}$$

where $(\xi_*(t,\varepsilon), y_*(t,\varepsilon))$ is a stable cycle of the auxiliary system

$$\begin{aligned} \dot{\xi} &= (1-\xi)\xi^2 - \xi y, \\ \varepsilon\dot{y} &= [(1-\alpha)\xi - \delta]y. \end{aligned} \tag{3.33}$$

Indeed, after the changes of notation $\xi \to x$, $\delta/(1-\alpha) \to \delta$, and $\varepsilon/(1-\alpha) \to \varepsilon$, system (3.33) is reduced to the form (1.4). Thus, under the above-imposed condition on the parameter δ (see (3.31)), by virtue of Theorems 1.2 and 2.1, it necessarily has the cycle indicated. Further, reasoning in a similar way, we conclude that for $d = 0$, $0 < \delta < \alpha/2$, system (3.2) admits a stable cycle

$$(\xi_1, \xi_2, y) = (\xi_*(t,\varepsilon), 0, y_*(t,\varepsilon)), \tag{3.34}$$

where $(\xi_*(t,\varepsilon), y_*(t,\varepsilon))$ is a stable cycle of the auxiliary system

$$\begin{aligned} \dot{\xi} &= (1-\xi)\xi^2 - \xi y, \\ \varepsilon\dot{y} &= [\alpha\xi - \delta]y. \end{aligned}$$

In the mapping (3.19), associated with the cycles (3.32) and (3.34) are the exponentially stable fixed points

$$\begin{aligned} (\xi_{1,0}, \xi_{2,0}) &= (0, x_0^*(\delta/(1-\alpha))), \\ (\xi_{1,0}, \xi_{2,0}) &= (x_0^*(\delta/\alpha), 0), \end{aligned} \tag{3.35}$$

where $x_0^* = x_0^*(\delta)$ is an exponentially stable fixed point of the mapping (1.17) (here we explicitly emphasize its dependence on the parameter δ). For small $d > 0$, stable fixed points close to (3.35) are preserved but shifted inside the cone $\mathbb{R}_+^2$. In the original system (3.2), they are associated with stable relaxation cycles transforming into the cycles (3.32) and (3.34) for $d = 0$. For $\alpha = 0.01$, $\delta = 0.3$, $d = 0.01$, and $\varepsilon = 0.01$, the t-dependences of the coordinates ξ_1, ξ_2, and y for one of such cycles are depicted in Fig. 3.1. The solid line in the figure indicates the graph of the function y; the dotted lines, of the functions ξ_1 and ξ_2 (the graph of the component ξ_1 lies below the graph of ξ_2).

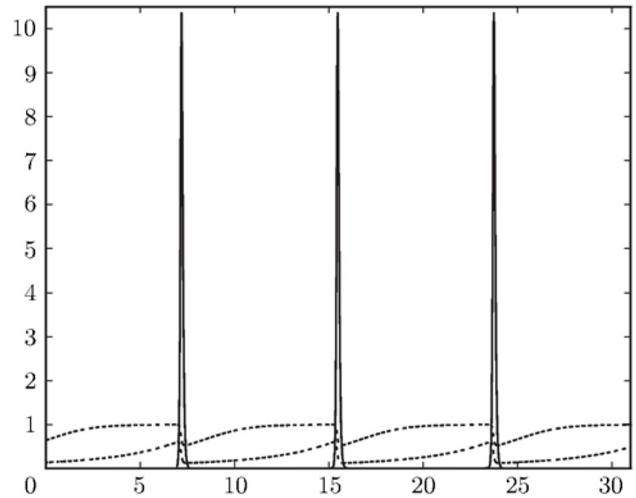

Fig. 3.1. Dependences of the coordinates ξ_1, ξ_2, and y on t for nonclassical relaxation cycle.

In conclusion, let us dwell on two problems still awaiting solution.

The first one is related to the analysis of the attractors of the multidimensional system

$$\dot{\xi}_j = (1-\xi_j)\xi_j^2 - \xi_j y + d(\xi_{j+1} - 2\xi_j + \xi_{j-1}),$$
$$\varepsilon\dot{y} = \left[\sum_{j=1}^{m}\alpha_j\xi_j - \delta\right]y, \quad j=1,\ldots,m, \quad \xi_0 = \xi_1, \quad \xi_{m+1} = \xi_m, \tag{3.36}$$

where $m \geq 2$, $d, \delta = \text{const} > 0$, $\alpha_j = \text{const} \in (0,1)$, $j = 1,\ldots,m$, and $\sum_{j=1}^{m}\alpha_j = 1$. Of interest are the questions of stability of the homogeneous cycle of system (3.36) and of the existence of stable cycles distinct from the homogeneous one in this system.

The second problem is to extend the theory of nonclassical relaxation oscillations to a more general mathematical model than (1.3),

$$\dot{N}_1 = r_1\left[1 - \frac{N_1}{K_1}\right]N_1^2 - aN_1N_2, \quad \dot{N}_2 = r_2\left[\frac{N_1}{K_1} - b - \frac{N_2}{K_2}\right]N_2, \tag{3.37}$$

where $r_1, r_2, K_1, K_2, a, b > 0$. It can readily be seen that, after appropriate normalizations, system (3.37) acquires a form similar to (1.4), namely,

$$\dot{x} = (1-x)x^2 - xy, \quad \varepsilon\dot{y} = (x - \delta - cy)y, \tag{3.38}$$

where $0 < \varepsilon \ll 1$ and $\delta, c = \text{const} > 0$. It can be shown that under the additional conditions $0 < \delta < 1$, $c = \varepsilon\alpha$, and $\alpha = \text{const} \in (0,1)$, analogs of Theorems 1.1 and 1.2 hold for system (3.38). However, the question about the existence of exponentially stable fixed points of the corresponding limit one-dimensional mapping remains open in this case.

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