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NONLINEAR PHYSICS AND MECHANICS

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Nonintegrability of the Problem of the Motion of an Ellipsoidal Body with a Fixed Point in a Flow of Particles

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The problem of the motion, in the free molecular flow of particles, of a rigid body with a fixed point bounded by the surface of an ellipsoid of revolution is considered. This problem is similar in many aspects to the classical problem of the motion of a heavy rigid body about a fixed point. In particular, this problem possesses the integrable cases corresponding to the classical Euler – Poinsot, Lagrange and Hess cases of integrability of the equations of motion of a heavy rigid body with a fixed point. A natural question arises about the existence of analogues of other integrable cases that exist in the problem of motion of a heavy rigid body with a fixed point (Kovalevskaya case, Goryachev – Chaplygin case, etc) for the system considered. Using the standard Euler angles as generalized coordinates, the Hamiltonian function of the system is derived. Equations of motion of the body in the flow of particles are presented in Hamiltonian form. Using the theorem on the Liouville-type nonintegrability of Hamiltonian systems near elliptic equilibrium positions, which has been proved by V. V. Kozlov, necessary conditions for the existence in the problem under consideration of an additional analytic first integral independent of the energy integral are presented. We have proved that the necessary conditions obtained are not fulfilled for a rigid body with a mass distribution corresponding to the classical Kovalevskaya integrable case in the problem of the motion of a heavy rigid body with a fixed point. Thus, we can conclude that this system does not possess an integrable case similar to the Kovalevskaya integrable case in the problem of the motion of a heavy rigid body with a fixed point.

Keywords: rigid body with a fixed point, free molecular flow of particles, Hamiltonian system, nonintegrability

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1. Introduction. V. V. Kozlov's theorem on the nonexistence of an analytic first integral near the equilibrium position of a Hamiltonian system

In 1976 V. V. Kozlov in his paper [1] (see also [2, 3]) proved a theorem which gives sufficient conditions of the nonexistence, for the Hamiltonian system, of a first integral analytic in canonical variables and independent of the Hamiltonian function H. Below we give a statement of the problem using the notations from [1] and a formulation of the corresponding theorem.

Let us consider the system of canonical equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n, \quad n \geqslant 2 \tag{1.1}$$

with the Hamiltonian function $H(y_1, \ldots, y_n, x_1, \ldots, x_n, \boldsymbol{\varepsilon})$ depending analytically on the variables $\boldsymbol{y} = (y_1, \ldots, y_n)$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and on the parameter $\boldsymbol{\varepsilon}$, which takes values in some connected domain $D \in \mathbb{R}^r$. Suppose that for all $\boldsymbol{\varepsilon}$ the point $y_i = 0$, $x_i = 0$ $(i = 1, \ldots, n)$ is an equilibrium position of the system (1.1). In the vicinity of an equilibrium position $y_i = 0$, $x_i = 0$ $(i = 1, \ldots, n)$ the Hamiltonian function H can be represented as follows:

$$H = H^{(2)} + H^{(3)} + \cdots,$$

where $H^{(s)}$ is a homogeneous form of degree s in $\boldsymbol{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$. The coefficients of this expansion are analytic functions of the parameter $\boldsymbol{\varepsilon}$. Let us assume that for all $\boldsymbol{\varepsilon} \in D$ the frequencies of linear oscillations $\boldsymbol{\omega}(\boldsymbol{\varepsilon}) = (\omega_1(\boldsymbol{\varepsilon}), \ldots, \omega_n(\boldsymbol{\varepsilon}))$ do not satisfy any resonant relation

$$(\boldsymbol{m} \cdot \boldsymbol{\omega}) = m_1\omega_1 + \cdots + m_n\omega_n = 0$$

of order $|m_1| + \cdots + |m_n| \leqslant m - 1$. Using Birkhoff's normalization method (see, for example, [4, 5]), we can find a canonical transformation $(\boldsymbol{y}, \boldsymbol{x}) \to (\boldsymbol{p}, \boldsymbol{q})$ such that in the new variables

$$H^{(2)} = \frac{1}{2}\sum_{i=1}^{n} \omega_i\rho_i, \quad H^{(k)} = H^{(k)}(\rho_1, \ldots, \rho_n, \boldsymbol{\varepsilon}), \quad k \leqslant m - 1,$$

where $\rho_i = p_i^2 + q_i^2$. The corresponding transformation is analytic in $\boldsymbol{\varepsilon}$. Now we introduce the canonical action–angle variables $(\boldsymbol{I}, \boldsymbol{\varphi})$ by the formulas

$$I_i = \frac{\rho_i}{2}, \quad \varphi_i = \arctan\frac{p_i}{q_i} \quad (1 \leqslant i \leqslant n).$$

In the canonical variables $(\boldsymbol{I}, \boldsymbol{\varphi})$ we have

$$H = H^{(2)}(\boldsymbol{I}, \boldsymbol{\varepsilon}) + \cdots + H^{(m-1)}(\boldsymbol{I}, \boldsymbol{\varepsilon}) + H^{(m)}(\boldsymbol{I}, \boldsymbol{\varphi}, \boldsymbol{\varepsilon}) + \cdots$$

We represent the trigonometric polynomial $H^{(m)}$ as a finite Fourier series

$$H^{(m)} = \sum h_{\boldsymbol{k}}^{(m)}(\boldsymbol{I}, \boldsymbol{\varepsilon}) \exp(i(\boldsymbol{k} \cdot \boldsymbol{\varphi})).$$

Theorem 1 (V. V. Kozlov [1–3]). *Let* $(\boldsymbol{k} \cdot \boldsymbol{\omega}(\boldsymbol{\varepsilon})) \not\equiv 0$ *for all* $\boldsymbol{k} \in \mathbb{Z}^n \setminus \boldsymbol{0}$. *Suppose that for some* $\boldsymbol{\varepsilon}_0 \in D$ *the resonant relation* $(\boldsymbol{k}_0 \cdot \boldsymbol{\omega}(\boldsymbol{\varepsilon}_0)) = 0$, $|\boldsymbol{k}_0| = m$ *is satisfied and* $h_{\boldsymbol{k}_0}^{(m)} \not\equiv 0$. *Then* *the canonical equations* (1.1) *with Hamiltonian function* $H = \sum H^{(s)}$ *do not have a complete set of* (*formal*) *integrals* $F_j = \sum F_j^{(s)}$, *whose quadratic terms* $F_j^{(2)}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\varepsilon})$ *are independent for all* $\boldsymbol{\varepsilon} \in D$.

Remark 1. Note that under the assumptions of V. V. Kozlov's Theorem 1 there may exist independent integrals with dependent (for certain values of $\boldsymbol{\varepsilon}$) quadratic parts of their Maclaurin expansions. Here is a simple example: the canonical equations with the Hamiltonian function

$$H = \frac{1}{2}\left(x_1^2 + y_1^2\right) + \frac{\alpha}{2}\left(x_2^2 + y_2^2\right) + 2x_1y_1y_2 - x_2y_1^2 + x_1^2x_2$$

have a first integral

$$F = x_1^2 + y_1^2 + 2\left(x_2^2 + y_2^2\right).$$

For $\alpha = 2$, it is dependent on the quadratic form $H^{(2)}$. However, all conditions of Theorem 1 are satisfied.

The advantage of V. V. Kozlov's Theorem 1 consists in the absence of preliminary restrictive assumptions regarding the parameters of the system. This advantage substantially compensates for the fact that the additional integral must belong to the class of analytic functions whose quadratic parts is functionally independent of the quadratic part of the Hamiltonian function.

V. V. Kozlov's Theorem 1 was successfully applied to prove the nonexistence of an additional first integral in the plane circular restricted three-body problem [1–3], to study the integrability of the problem of motion about a fixed point of a dynamically symmetric rigid body with the center of mass lying in the equatorial plane of the ellipsoid of inertia [1, 3, 6], to prove the nonexistence of an additional integral in the problem of the motion of a heavy double plane pendulum [6–8], to obtain necessary conditions for the existence of an additional first integral in the problem of the motion of a dynamically symmetric ellipsoid on a smooth horizontal plane [9], and to study nonintegrability of the Kirchhoff equations of motion of a rigid body in a fluid [10, 11].

In this paper V V. Kozlov's Theorem 1 is used to derive necessary conditions for the existence of an additional analytic integral in the problem of motion in the flow of particles of a rigid body with a fixed point bounded by the surface of an ellipsoid of revolution.

2. Formulation of the problem. Hamiltonian function of the problem

Equations of motion of a rigid body with a fixed point, bounded by the surface of an ellipsoid and exposed to the flow of particles, have the form [12, 13]

$$\begin{aligned}
A_1\dot{\omega}_1 + (A_3 - A_2)\omega_2\omega_3 &= \rho v_0^2 \pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}(h_2\gamma_3 - h_3\gamma_2),\\
A_2\dot{\omega}_2 + (A_1 - A_3)\omega_1\omega_3 &= \rho v_0^2 \pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}(h_3\gamma_1 - h_1\gamma_3),\\
A_3\dot{\omega}_3 + (A_2 - A_1)\omega_1\omega_2 &= \rho v_0^2 \pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}(h_1\gamma_2 - h_2\gamma_1);\\
\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot{\gamma}_2 &= \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2.
\end{aligned} \tag{2.1}$$

Here A_1, A_2 and A_3 are the moments of inertia of the body about the principal axes of inertia $Ox_1x_2x_3$ with origin at the fixed point O, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector of the body, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector directed along the flow of particles, ρ is the constant density of the flow of particles, v_0 is the constant velocity of particles in the flow, a_1, a_2 and a_3 are the lengths of the semiaxes of the ellipsoid bounding the rigid body, and $\boldsymbol{h} = (h_1, h_2, h_3)$ is the vector directed from a fixed point to the center of the ellipsoid bounding the rigid body.

For any parameter values, Eqs. (2.1) possess the first integrals

$$J_1 = A_1\omega_1\gamma_1 + A_2\omega_2\gamma_2 + A_3\omega_3\gamma_3 = c_1 = \text{const}, \quad J_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{2.2}$$

Let us assume that the center of the ellipsoid lies on the first principal axis of inertia Ox_1 with origin at the fixed point O, at a distance l from the fixed point. In other words, in Eqs. (2.1) we put

$$h_1 = l, \quad h_2 = 0, \quad h_3 = 0.$$

We also assume that the ellipsoid bounding the rigid body is an ellipsoid of revolution with the axis of symmetry passing through the fixed point O. Therefore, in Eqs. (2.1) we put

$$a_1 = b, \quad a_2 = a_3 = a.$$

In addition, we assume that the body is dynamically symmetric and that the axis of dynamical symmetry of the body does not coincide with the axis of symmetry of the ellipsoid bounding the body. In other words, we assume that

$$A_1 = A_2 = A, \quad A_3 = C.$$

Then the equations of motion in the flow of particles of a rigid body with a fixed point bounded by the surface of an ellipsoid of revolution will be rewritten as follows:

$$\begin{gathered} A\dot{\omega}_1 + (C - A)\omega_2\omega_3 = 0, \\ A\dot{\omega}_2 + (A - C)\omega_1\omega_3 = -\rho v_0^2 \pi a^2 bl\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}\gamma_3, \\ C\dot{\omega}_3 = \rho v_0^2 \pi a^2 bl\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}\gamma_2; \\ \dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2. \end{gathered} \tag{2.3}$$

In addition to the first integrals (2.2), Eqs. (2.3) have a first integral of energy integral type

$$H = \frac{A}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{C}{2}\omega_3^2 - G(\gamma_1) = h = \text{const}.$$

The function $G(\gamma_1)$ is written differently depending on whether the ellipsoid bounding the rigid body is prolate ($b > a$) or oblate ($a > b$). For a prolate ellipsoid of revolution ($b > a$), the function $G(\gamma_1)$ has the form

$$G(\gamma_1) = \frac{\rho v_0^2 \pi a^2 bl}{2}\gamma_1\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}} + \frac{\rho v_0^2 \pi bl}{2\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}}\arctan\left(\frac{\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}\gamma_1}{\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}}\right).$$

For an oblate ellipsoid of revolution ($a > b$), the function $G(\gamma_1)$ has the form

$$G(\gamma_1) = \frac{\rho v_0^2 \pi a^2 bl}{2}\gamma_1\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}} + \frac{\rho v_0^2 \pi bl}{2\sqrt{\frac{1}{b^2} - \frac{1}{a^2}}}\ln\left(a\sqrt{\frac{1}{b^2} - \frac{1}{a^2}}\gamma_1 + a\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}\right).$$

Further, we will consider the case of a prolate ellipsoid of revolution (the case of an oblate ellipsoid of revolution is considered in a similar way and gives the same result). As generalized coordinates in this problem we introduce the standard Euler angles θ, ψ and φ. Then we have

$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

and the Hamiltonian function of the problem in standard notation has the form

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{A} + \frac{p_\varphi^2}{C} + \frac{(p_\psi - p_\varphi\cos\theta)^2}{A\sin^2\theta}\right) - \frac{\rho v_0^2\pi a^2 bl}{2}\sin\theta\sin\varphi\sqrt{\frac{1-\sin^2\theta\sin^2\varphi}{a^2} + \frac{\sin^2\theta\sin^2\varphi}{b^2}} -$$
$$- \frac{\rho v_0^2\pi bl}{2\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}}\arctan\left(\frac{\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}\sin\theta\sin\varphi}{\sqrt{\frac{1-\sin^2\theta\sin^2\varphi}{a^2} + \frac{\sin^2\theta\sin^2\varphi}{b^2}}}\right). \quad (2.4)$$

Obviously, the Hamiltonian function H does not depend on the generalized coordinate ψ, that is, the generalized momentum p_ψ is a constant. The generalized momentum p_ψ is the area integral J_1 (see (2.2)). The equations of motion of the body have a Hamiltonian form with the Hamiltonian function (2.4), in which p_ψ is a parameter. We will assume that the parameter p_ψ is the parameter that was mentioned in the statement of V. V. Kozlov's Theorem 1. Let us obtain the necessary conditions for the existence of an additional first integral analytic in p_ψ and independent of the Hamiltonian function H.

3. Application of V. V. Kozlov's Theorem 1

For any value of p_ψ the point

$$(p_\theta,\, p_\varphi,\, \theta,\, \varphi) = \left(0,\, 0,\, \frac{\pi}{2},\, \frac{\pi}{2}\right)$$

is the equilibrium of the Hamiltonian system considered. We denote

$$p_\theta = y_1, \quad p_\varphi = y_2, \quad \theta = \frac{\pi}{2} + x_1, \quad \varphi = \frac{\pi}{2} + x_2.$$

The units of measurement can always be chosen so that

$$\pi\rho v_0^2 l a^2 = 1, \quad A = 1.$$

We introduce also the following parameters:

$$p_\psi = \sqrt{x}, \quad \frac{1}{C} = y, \quad \frac{b^2}{a^2} = z.$$

In a neighborhood of the equilibrium point $y_1 = 0$, $y_2 = 0$, $x_1 = 0$, $x_2 = 0$ the expansion of the Hamiltonian (2.4) has the form

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \cdots,$$
$$H^{(2)}(y_1,\, y_2,\, x_1,\, x_2) = \frac{1}{2}y_1^2 + \frac{y}{2}y_2^2 + \sqrt{x}x_1y_2 + \frac{(1+x)}{2}x_1^2 + \frac{1}{2}x_2^2, \quad H^{(3)}(y_1,\, y_2,\, x_1,\, x_2) = 0,$$
$$H^{(4)}(y_1,\, y_2,\, x_1,\, x_2) = \frac{1}{2}x_1^2y_2^2 + \frac{5}{6}\sqrt{x}x_1^3y_2 + \left(\frac{z}{4} - \frac{1}{2}\right)x_1^2x_2^2 + \left(\frac{x}{3} + \frac{z}{8} - \frac{1}{6}\right)x_1^4 + \left(\frac{z}{8} - \frac{1}{6}\right)x_2^4.$$

Note that in the case of $z = 1$, i. e., when the rigid body is bounded by the sphere, the expressions $H^{(2)}(y_1, y_2, x_1, x_2)$ and $H^{(4)}(y_1, y_2, x_1, x_2)$ exactly coincide with the corresponding expressions obtained by V. V. Kozlov [1–3] when studying the problem of motion of a heavy dynamically symmetric rigid body with a fixed point, with the center of mass situated in the equatorial plane of the ellipsoid of inertia.

The characteristic equation of the linear system with the Hamiltonian function $H = H^{(2)}$ is written as follows:

$$\lambda^4 + (1 + x + y)\lambda^2 + y(1 + x) - x = 0. \tag{3.1}$$

Obviously, the roots of the characteristic equation are purely imaginary if

$$y > \frac{x}{1 + x}.$$

Let E denote the subset of $\mathbb{R}^2_+$, where this inequality is satisfied. The ratio of the frequencies $\frac{\lambda_1}{\lambda_2} = 3$ if the parameters x and y satisfy the equation

$$9x^2 - 82xy + 9y^2 + 118x - 82y + 9 = 0. \tag{3.2}$$

This is the equation of a hyperbola; for $x > 0$ and $y > 0$ its branches are entirely in E.

From the triangle inequality for the moments of inertia ($A_1 + A_2 \geqslant A_3$) it follows that $y \geqslant \frac{1}{2}$. For any fixed $y_0 \geqslant \frac{1}{2}$, there exists $x_0 > 0$ such that the point (x_0, y_0) satisfies Eq. (3.2). Consider a small interval (a, b) of variation of the parameter x, including the point x_0. For $x \in (a, b)$, $y = y_0$ the roots of the characteristic equation are purely imaginary and distinct. When $x = x_0$, the frequencies λ_1 and λ_2 are related by the equation $\lambda_1 - 3\lambda_2 = 0$. It remains to find out when the secular coefficient $h^{(4)}_{1,-3}$ is zero.

After a linear canonical transformation $(y_1, y_2, x_1, x_2) \to (p_1, p_2, q_1, q_2)$ by the formulas

$$y_1 = \frac{1}{1 + \alpha^2} p_1 + \frac{\alpha^2}{1 + \alpha^2} q_2, \quad y_2 = \frac{1}{\alpha} p_2 + \alpha q_1, \quad x_1 = q_1 - p_2, \quad x_2 = \frac{\alpha}{1 + \alpha^2}(q_2 - p_1),$$
$$\sqrt{x}\alpha^2 + (x + 1 - y)\alpha - \sqrt{x} = 0,$$

the quadratic part $H^{(2)}$ of the Hamiltonian function H is represented as follows:

$$H^{(2)} = \frac{B_1}{2} p_1^2 + \frac{K_1}{2} q_1^2 + \frac{B_2}{2} p_2^2 + \frac{K_2}{2} q_2^2,$$
$$B_1 = \frac{1}{1 + \alpha^2}, \quad B_2 = \frac{y - 2\alpha\sqrt{x} + (1 + x)\alpha^2}{\alpha^2} = \frac{(1 + \alpha^2)(y - \alpha\sqrt{x})}{\alpha^2},$$
$$K_1 = \alpha^2 y + 2\alpha\sqrt{x} + 1 + x = (1 + \alpha^2)\left(y + \frac{\sqrt{x}}{\alpha}\right), \quad K_2 = \frac{\alpha^2}{1 + \alpha^2}.$$

Now we introduce action–angle variables $(\boldsymbol{I}, \boldsymbol{\varphi})$ by the formulas

$$q_1 = i\sqrt{\frac{I_1}{2}\sqrt{\frac{B_1}{K_1}}}(\exp(-i\varphi_1) - \exp(i\varphi_1)), \quad p_1 = \sqrt{\frac{I_1}{2}\sqrt{\frac{K_1}{B_1}}}(\exp(i\varphi_1) + \exp(-i\varphi_1)),$$
$$q_2 = i\sqrt{\frac{I_2}{2}\sqrt{\frac{B_2}{K_2}}}(\exp(-i\varphi_2) - \exp(i\varphi_2)), \quad p_2 = \sqrt{\frac{I_2}{2}\sqrt{\frac{K_2}{B_2}}}(\exp(i\varphi_2) + \exp(-i\varphi_2)).$$

Here i is the unit imaginary number. In the new variables the form $H^{(4)}$ will be written as follows:

$$H^{(4)} = \sum_{0\leqslant|m_1|+|m_2|\leqslant 4} h^{(4)}_{m_1,m_2} \exp(i(m_1\varphi_1 + m_2\varphi_2)).$$

The condition for vanishing of the coefficient $h^{(4)}_{1,-3}$ in the expansion of the function $H^{(4)}$ can be reduced to the following form:

$$\begin{gathered}27x^3z + 111x^2yz - 159xy^2z - 243y^3z - 9x^3 - 617x^2y - 39x^2z + 2093xy^2 - 118xyz + 1701y^3 + \\ + 621y^2z + 653x^2 - 4374xy - 59xz - 2727y^2 - 129yz + 2633x + 543y + 7z - 29 = 0. \quad (3.3)\end{gathered}$$

Thus, the following theorem holds.

Theorem 2. *Necessary conditions on the parameters y and z for the existence of an additional integral analytic in canonical variables and the parameter x and independent of the Hamiltonian function H in the problem of motion in the flow of particles of a dynamically symmetric rigid body with a fixed point bounded by the surface of an ellipsoid of revolution whose center lies in the equatorial plane of the ellipsoid of inertia can be found as the resultant of Eqs.* (3.2) *and* (3.3).

Remark 2. For $z = 1$, i.e., in the case where the rigid body is bounded by a sphere, Eqs. (3.2) and (3.3) take the form

$$9x^2 - 82xy + 9y^2 + 118x - 82y + 9 = 0, \quad (3.4)$$

$$18x^3 - 506x^2y + 1934xy^2 + 1458y^3 + 614x^2 - 4492xy - 2106y^2 + 2574x + 414y - 22 = 0, \quad (3.5)$$

and exactly coincide with the necessary conditions for the existence of an additional integral in the problem of motion of a heavy dynamically symmetric rigid body with a fixed point and with the center of mass situated in the equatorial plane of the ellipsoid of inertia, obtained by V. V. Kozlov [1–3, 6]. The algebraic curves (3.4) and (3.5) intersect at two points (x, y):

$$\left(\frac{4}{3}, 1\right) \quad \text{and} \quad (7, 2),$$

which correspond to the Lagrange integrable case ($A = C$) and the Kovalevskaya integrable case ($A = 2C$).

Let us put in conditions (3.2) and (3.3) $y = 2$, i.e., consider a rigid body with a mass distribution corresponding to the Kovalevskaya integrable case in the problem of motion of a heavy rigid body with a fixed point. Then condition (3.2) takes the form

$$(9x + 17)(x - 7) = 0,$$

and can only be valid if $x = 7$. Substituting the values $x = 7$ and $y = 2$ into condition (3.3) gives

$$12\,000(z - 1) = 0.$$

Thus, we have the following statement.

Corollary 1. *For a rigid body with a mass distribution corresponding to the Kovalevskaya case, an additional first integral independent of the energy integral can exist only when the rigid body is bounded by a sphere. In the case where a rigid body exposed to the flow of particles is bounded by the ellipsoid, there is no additional first integral.*

Analysis of Eqs. (3.2) and (3.3), performed using MAPLE 7 symbolic computations software, shows that this system has solutions

$$x = 0, \quad y = \frac{1}{9}; \qquad x = -\frac{16}{3}, \quad y = 1; \qquad x = \frac{4}{3}, \quad y = 1 \tag{3.6}$$

existing for any value of the parameter z. The first two of the solutions (3.6) do not satisfy the conditions

$$x > 0, \quad y \geqslant \frac{1}{2}$$

and therefore they have no physical meaning. As for the third solution, it corresponds to the Lagrange integrable case ($A = C$). Thus, in this problem, for any shape of the ellipsoid (both when it is prolate and when it is oblate), there is an integrable case corresponding to the Lagrange case.

In addition to the three solutions (3.6), Eqs. (3.2) and (3.3) admit a z-dependent solution in which y is a root of the quadratic equation with coefficients depending on z, and x is expressed in terms of y and z:

$$(3z - 4)(7z - 52)y^2 - (76z^2 - 632z + 736)y + 20z^2 - 432z + 592 = 0,$$

$$x = \frac{\left(4048z^2 - 471z^3 - 3200 - 2672z\right) y + 3252z^2 - 54z^3 - 17\,424z + 18\,816}{2(3z - 4)(7z - 52)((23z - 32)y - 38z + 56)}.$$

Among the parameters (x, y, z) that belong to this solution, one can find parameters that have a physical meaning. These are, for example, the parameters

$$x = \frac{57}{23}, \quad y = \frac{30}{23}, \quad z = \frac{1}{5}.$$

Thus, for some parameter values, the necessary conditions for the existence of an additional first integral in the problem of motion of a rigid body with a fixed point in the flow of particles are satisfied. The study of existence of an additional first integral for such parameter values is a problem which we will try to investigate in the future.

4. Conclusions

In this paper we have presented necessary conditions for the existence of an additional analytic first integral independent of the energy integral in the problem of motion of a rigid body with a fixed point in the flow of particles. The necessary conditions obtained are always fulfilled in the case of motion of a dynamically symmetric rigid body with the center of mass lying on the axis of dynamical symmetry of the body (the case similar to the Lagrange integrable case of the classical problem of motion of a heavy rigid body with a fixed point) and these conditions are not fulfilled for the dynamically symmetric rigid body with the center of mass lying in the equatorial plane of the ellipsoid of inertia (the mass distribution similar to the Kovalevskaya integrable case in the classical problem of motion of a heavy rigid body with a fixed point). Thereby, we have proved the nonexistence of the integrable case similar to the Kovalevskaya integrable case in the problem of motion in the flow of particles of a rigid body with a fixed point.

Conflict of interest

The authors declare that they have no conflict of interest.

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