Dynamics and exact solutions of evolutionary partial differential systems

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By systems of evolutionary differential equations we mean systems of the form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \dots, \frac{\partial^k \mathbf{u}}{\partial \mathbf{x}^k}\right).$$

Here $\mathbf{x} = (x_1, \dots, x_n)$ is a vector of independent spatial variables, t is time, $\mathbf{u} = (u^1, \dots, u^m)$ and $\mathbf{f} = (f^1, \dots, f^m)$ are vector functions. We suppose that the functions $f_1, \dots, f_m$ belongs to the class C^∞ within its domain. The symbol $\partial^i \mathbf{u}/\partial \mathbf{x}^i$ $(i = 1, \dots, k)$ means the set of all partial derivatives of order i by $\mathbf{x}$.

The main idea is as follows.

This system generates a flow on maximal integral manifolds of some completely integrable distributions P [1,2], i.e. its right parts defines Lie algebra of symmetries of P. Consider the case when the distribution is generated by some overdetermined system of partial differential equations

$$\frac{\partial^{q+1}\mathbf{v}}{\partial \mathbf{x}^{\sigma+1_i}} = \mathbf{V}_{\sigma+1_i}\left(\mathbf{x}, \mathbf{v}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \dots, \frac{\partial^q \mathbf{v}}{\partial \mathbf{x}^q}\right), \quad |\sigma| = \sigma_1 + \dots + \sigma_n = q; i = 1, \dots n,$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a multi-index, $\sigma_i \in \{0, 1, \dots, q\}$, $|\sigma| = \sigma_1 + \dots + \sigma_n$, $\sigma + 1_i = (\sigma_1, \dots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \dots, \sigma_n)$ $\mathbf{v}$ is a vector-valued function of $\mathbf{x} = (x_1, \dots, x_n)$.

Let S be a shuffling symmetry of the distribution P [3]. There are a unique set of functions $\varphi^1, \dots, \varphi^m$ on J^q such that

$$S = \sum_{j=1}^{m} \varphi^j \frac{\partial}{\partial v^j_o} + \sum_{\substack{|\sigma|=1\\ j=1,\dots,m}} \mathcal{D}^\sigma(\varphi^j) \frac{\partial}{\partial v^j_\sigma} + \dots + \sum_{\substack{|\sigma|=q\\ j=1,\dots,m}} \mathcal{D}^\sigma(\varphi^j) \frac{\partial}{\partial v^j_\sigma}.$$

Here $o = (0, \dots, 0)$ is zero multi-index, $\mathcal{D}^\sigma = \mathcal{D}_1^{\sigma_1} \circ \dots \circ \mathcal{D}_n^{\sigma_n}$, and $\mathcal{D}_i^s$ is the s-th degree of the operator

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \sum_{\substack{0\le|\sigma|\le q\\ j=1,\dots,m}} v^j_{\sigma+1_i} \frac{\partial}{\partial v^j_\sigma} + \sum_{\substack{0\le|\sigma|=q\\ j=1,\dots,m}} V^j_{\sigma+1_i}(\mathbf{x}, \mathbf{v}_\sigma) \frac{\partial}{\partial v^j_\sigma} \quad (i = 1, \dots n).$$

Note that the distribution P is generated by the vector fields $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

The functions $\varphi^1, \ldots, \varphi^m$ satisfy the following system:

$$\mathcal{D}^{\sigma+1_i}(\varphi^j) - \sum_{s=1}^{n} \sum_{|\mu|=0}^{q} \mathcal{D}^{\mu}(\varphi^s) \frac{\partial V^j_{\sigma+1_i}}{\partial v^s_\mu} = 0, \quad i = 1, \ldots, n; \quad j = 1, \ldots, m.$$

Solving this system we can find the vector field S. Shifts along this vector field of solutions of the overdetermined system, we obtain a solution to the evolutionary system.

This method will be illustrated using the examples of the Boussinesq equation [4]

$$\begin{cases} u_t = u_{xx} + 2v_x, \\ v_t = -v_{xx} + 2uu_x - 2u_y. \end{cases}$$

It made it possible to construct a family of exact solutions of the Boussinesq equation which depends on six arbitrary parameters and one arbitrary function.

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