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Neutrino Mass in Effective Field Theory

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Abstract—In this review, the seesaw mechanism for generating the mass of active light neutrinos (both Majorana and Dirac) is considered on the basis of effective field theory. In particular, we review certain models that extend the Standard Model by introducing heavy sterile neutrinos and discuss the corresponding mechanisms for generating small masses of active neutrinos. Two appendices briefly describe the properties of Weyl, Dirac, and Majorana spinors in four dimensions and interrelations between such spinors. The third Appendix provides a simple proof of the theorem on Takagi diagonalization of the mass matrix for Majorana fermions.

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INTRODUCTION

In the minimal Standard Model (SM), neutrinos are massless left-chiral (left-handed) fermions [1] forming, together with the left-handed components of charged leptons, three doublets that are transformed according to the fundamental representation of the gauge group $SU(2)_L$. However, the discovery of neutrino oscillations showed that neutrino masses are non-zero, but for the three active neutrinos they are very small (see review [2]). From the data of oscillations, only two differences of squared neutrino masses (in the 3-neutrino mixing scheme) [2] are determined but not their absolute values, which allows us to obtain a lower limit on the largest of the three masses:

$$m_3 > \sqrt{\Delta m_{31}^2} \simeq 0.05 \text{ eV}. \quad (1)$$

The most stringent upper limit on the sum of light neutrino masses from modern cosmological data is [3]:

$$\sum m_\nu \equiv \sum_{i=1}^{3} m_i < 0.09 \text{ eV}. \quad (2)$$

In the minimal SM, the masses of fermions (charged leptons and quarks) are generated due to the Yukawa interaction of the Higgs scalar doublet φ with doublets of left-handed fermion components and right-handed fermion singlets, but neutrinos, being only left-handed, remain massless. For the generation of Dirac neutrino masses, which already makes it possible to describe neutrino oscillations, it is sufficient to introduce right-handed components of neutrino fields $\nu_{\alpha R}$ $(\alpha = e, \mu, \tau)$ and use the same Brout–Engler–Higgs mechanism as for charged fermions. However, it should be emphasized that right-handed neutrinos are fundamentally different from left-handed neutrinos and right-handed charged fermions. Namely, $\nu_{\alpha R}$ are sterile (unlike active $\nu_{\alpha L}$), i.e. they do not participate in electroweak (and, of course, strong) interactions, since their weak isospin and hypercharge are zero (fields $\nu_{\alpha R}$ are singlets of the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$).

The Lagrangian of the interaction generating Dirac neutrino masses has the form (we follow [4])

$$\mathcal{L}_{\nu H} = -y^{\nu}_{\alpha\beta} \bar{L}'_{\alpha L} \tilde{\varphi} \nu'_{\beta R} - \text{H.c.} \quad (3)$$

Here $y^{\nu}_{\alpha\beta}$ are complex Yukawa coupling constants, indices $\alpha, \beta = e, \mu, \tau$ enumerate lepton generations, and $\tilde{\varphi} = i\sigma_2 \varphi^{+T}$ (where σ_2 is the Pauli matrix, see (A.4)) is the doublet charge conjugated to the Higgs doublet

$$\varphi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad L'_{\alpha L} = \begin{pmatrix} \nu'_{\alpha L} \\ \ell'_{\alpha L} \end{pmatrix}, \quad (4)$$

with primes indicating fields with indefinite masses.

After spontaneous breaking of the gauge symmetry, a nonzero vacuum expectation value of the Higgs field arises, so that in the unitary gauge we have

$$\langle 0|\tilde{\varphi}|0\rangle = (v/\sqrt{2}, 0)^T, \quad v = (\sqrt{2} G_F)^{-1/2} \simeq 246 \,\text{GeV}, \quad (5)$$

and Lagrangian (3) takes the form (in matrix notation):

$$\mathcal{L}_{\ell H} = -\frac{1}{\sqrt{2}}(v + H)\bar{\nu}'_L Y^\nu \nu'_R - \text{H.c.}, \quad (6)$$

where H is the scalar Higgs field, $Y^\nu = (y^\nu_{\alpha\beta})$ denotes the matrix of the Yukawa couplings, $\nu'_P = (\nu'_{eP}, \nu'_{\mu P}, \nu'_{\tau P})^T$, $P = L, R$ (see Eq. (A.9) in Appendix A).

After the bi-unitary diagonalization of the matrix Y^ν,

$$\frac{v}{\sqrt{2}}(V^\nu_L)^+ Y^\ell V^\nu_R = \frac{v}{\sqrt{2}}\text{diag}(y_1, y_2, y_3) = \text{diag}(m_1, m_2, m_3) \equiv M^\nu, \quad (7)$$

we obtain the Lagrangian

$$\mathcal{L}_{\nu H} = -\left(1 + \frac{H}{v}\right)\left(\bar{n}_L M^\nu n_R + \text{H.c.}\right) = -\left(1 + \frac{H}{v}\right)\sum_{k=1}^{3} m_k \bar{\nu}_k \nu_k, \quad (8)$$

in terms of physical fields (with definite masses)

$$n_L = (V^\nu_L)^+ \nu'_L = (\nu_{1L}, \nu_{2L}, \nu_{3L})^T, \quad n_R = (V^\nu_R)^+ \nu'_R = (\nu_{1R}, \nu_{2R}, \nu_{3R})^T. \quad (9)$$

The Lagrangian (8) includes the mass terms (here $\nu_k = \nu_{kL} + \nu_{kR}$ are the four-component Dirac fields) and the term of the interaction of massive neutrinos with the Higgs boson, and what is more, the neutrino masses expressed through the Yukawa couplings y_k and the vacuum condensate v (see (7)):

$$m_k = y_k \frac{v}{\sqrt{2}}. \quad (10)$$

The lepton weak charged current, which describes the interaction with W bosons, includes fields of the left-handed neutrinos with definite flavors $\nu_L = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^T$ which are superpositions of the left-handed neutrino fields with definite masses (see (9)):

$$j^{(-)}_\lambda = \bar{\ell}'_L \gamma_\lambda \nu'_L = \bar{\ell}_L (V^\ell_L)^+ \gamma_\lambda V^\nu_L n_L = \bar{\ell}_L \gamma_\lambda \nu_L = \sum_{\alpha = e,\mu,\tau} \bar{\ell}_{\alpha L} \gamma_\lambda \nu_{\alpha L}. \quad (11)$$

Here

$$\nu_L = U n_L, \quad \nu_{\alpha L} = \sum_{k=1}^{3} U_{\alpha k} \nu_{kL}, \quad (12)$$

and $U = \|U_{\alpha k}\| = (V^\ell_L)^+ V^\nu_L$ is the unitary Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix of lepton mixing, for which, in the case of Dirac neutrinos, the standard parametrization[1] is

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (13)$$

where we use the concise notation: $s_{ij} := \sin\theta_{ij}$ and $c_{ij} := \cos\theta_{ij}$.

We emphasize that the Standard Model determines only the general structure of the matrix (13) but does not predict the numerical values of its parameters, which are determined from experimental data on neutrino oscillations (see [2, 5–7]). For definiteness, we present data from [7], corresponding to the best fit and two options of the hierarchy of the neutrino mass spectrum (see [2, 4]), normal and inverted (indicated in parentheses):

$$s^2_{12}/10^{-1} = 3.18 \pm 0.16 (3.18 \pm 0.16),$$
$$s^2_{23}/10^{-1} = 5.74 \pm 0.14 (5.78^{+0.10}_{-0.17}),$$
$$s^2_{13}/10^{-2} = 2.200^{+0.069}_{-0.062} (2.225^{+0.064}_{-0.070}),$$
$$\delta/\pi = 1.08^{+0.13}_{-0.12} (1.58^{+0.15}_{-0.16}).$$

As it can be seen, a reliable value of the CP-violating phase δ cannot yet be extracted from modern experimental data.

Equation (10) is applicable for any Dirac fermions. From this it follows that the Yukawa coupling constant y_f of the fermion with the Higgs boson increases with increasing mass of the fermion, and experimental data [2] demonstrate a huge hierarchy of the mass spectrum of fundamental fermions and corresponding y_f. So taking into account (5), for the neutrino (we choose its mass $m_\nu \simeq 0.05$ eV, see (1)), electron and t-quark we obtain:

$$y_\nu \simeq 3 \times 10^{-13}, \quad y_e \simeq 3 \times 10^{-6}, \quad y_t \simeq 1. \quad (14)$$

This hierarchy is one of the fundamental problems of the elementary particle physics, which cannot be solved within the framework of the Standard Model and requires its extension [8–11]. To study the effects of new physics not described by the Standard Model, the concept of effective field theory (EFT) [14–17] developed by S. Weinberg [12, 13] is used.

Under the assumption that the energy scale Λ of new physics is significantly larger than the character-

[1] In the case of the Majorana neutrinos, the PMNS-matrix (13) is modified [2], $U \to U\text{diag}(e^{i\eta_1}, e^{i\eta_2}, 1)$, and depends on three CP-violating phases δ, η_1, η_2.

istic SM scale v (see (5)), the EFT Lagrangian is represented as

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{SM}} + \sum_{n>4}\Lambda^{-n}\sum_{k}C_k^{(n)}\mathbb{O}_k^{(n)}. \quad (15)$$

Here $\mathscr{L}_{\text{SM}}$ is the SM Lagrangian, and the remaining terms describe the effects of new physics and include composite operators $\mathbb{O}_k^{(n)}$ with mass dimension $n = 5,6,\ldots$; $C_k^{(n)}$ are the numerical dimensionless parameters. The operators $\mathbb{O}_k^{(n)}$ are invariant under the SM gauge group $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ and are composed only of the SM fields. The coeffcients $C_k^{(n)}$ are determined from experimental data or, if the Lagrangian of a particular theory extending the SM is known, $C_k^{(n)}$are expressed in terms of coupling constants and masses of new (heavy) particles by matching the amplitudes of physical processes obtained on the basis of the two specified Lagrangians (in the region of relatively low energies $E \ll \Lambda$, where the effective Lagrangian (15) is applicable). The Lagrangian (15) can also be defined as the Lagrangian for the effective action obtained from the generating functional of the extended theory by integration over "heavy" fields [14–17].

There is a unique set of operators $\mathbb{O}^{(5)}$ of dimension 5, composed of the SM fields and possessing gauge symmetry [18]:

$$\mathbb{O}^{(5)} = z_{\alpha\beta}\left(\overline{L}'_{\alpha\mathrm{L}}\tilde{\varphi}\right)\left(\tilde{\varphi}^T L'^{c}_{\beta\mathrm{L}}\right) + \mathrm{H.c.} \quad (16)$$

Here $z_{\alpha\beta} = z_{\beta\alpha}$ is a set of (complex) dimensionless constants, $L'^{c}_{\beta\mathrm{L}} = C\overline{L}'^{T}_{\beta\mathrm{L}}$ is the charge conjugated doublet and C is the charge conjugation operator (see Eq. (A.14) in Appendix A). We stress that the operator $\mathbb{O}^{(5)}$, which is absent in the SM Lagrangian, does not preserve the total lepton number, changing it by two units. After spontaneous symmetry breaking, this operator generates a mass term for neutrinos

$$\mathscr{L}_{\nu\mathrm{M}} = -\frac{1}{2}M'_{\alpha\beta}\overline{\nu}'_{\alpha\mathrm{L}}\nu'^{c}_{\beta\mathrm{L}} - \mathrm{H.c.}, \quad M'_{\alpha\beta} = z_{\alpha\beta}\frac{v^2}{\Lambda}. \quad (17)$$

The symmetric (complex) mass matrix M' is transformed to the diagonal form by using the unitary matrix V (see [4], as well as the comment on Eq. (46) below):

$$V^T M' V = M = \mathrm{diag}(m_1, m_2, m_3), \quad (18)$$

where m_k are positive numbers, and the initial left-handed flavor fields are represented in the form of left-handed field components with definite masses:

$$\nu'_\mathrm{L} = V n_\mathrm{L}, \quad n_\mathrm{L} = (\nu_{1\mathrm{L}}, \nu_{2\mathrm{L}}\nu_{3\mathrm{L}})^T. \quad (19)$$

Making use of Eqs. (18) and (19), we reduce the mass term (17) to the diagonal form

$$\mathscr{L}_{\nu M} = -\frac{1}{2}\left(\overline{n}_\mathrm{L} M n_\mathrm{L}^c + \overline{n}_\mathrm{L}^c M n_\mathrm{L}\right) = -\frac{1}{2}\sum_{k=1}^{3} m_k \overline{\nu}_k \nu_k, \quad (20)$$

where

$$\nu_k = \nu_{k\mathrm{L}} + \nu_{k\mathrm{L}}^c = \nu_k^c. \quad (21)$$

Thus, massive neutrinos turn out to be Majorana particles (see Appendix A) coinciding with their antiparticles. As it follows from (17) and (18), their masses

$$m_k = z_k\frac{v^2}{\Lambda}, \quad (22)$$

are significantly less than the masses of charged leptons (see (10)) due to the presence of the suppressing factor v/Λ, caused by the effects of new physics. The typical neutrino mass scale can be represented as

$$m_\nu \sim \frac{v^2}{\Lambda} \simeq 6\times 10^{-2}\left(\frac{10^{15}\ \mathrm{GeV}}{\Lambda}\right)\mathrm{eV}, \quad (23)$$

where 10^{15} GeV is the typical energy scale for the grand unified models.

In this paper, we review a number of models that extend the SM by introducing heavy sterile neutrinos and discuss the corresponding mechanisms for the generation of small masses of active neutrinos.

1. SEESAW MECHANISM FOR GENERATING NEUTRINO MASSES

To explain the small masses of active neutrinos, a seesaw mechanism (SSM) of their generation was proposed, which is caused by the interaction of flavor neutrinos with heavy right-handed Majorana neutrinos [19–23]. There are three types of SSM which are classified in [24] (see also [25]).

In this paper, we will limit ourselves to considering SSM of type I, which is based on expanding the SM by adding three heavy right-handed neutrinos (singlets of the gauge group $\mathrm{SU}(2)_L$) while preserving the standard Higgs doublet. For the SSM of type II, a heavy Higgs triplet is added; for type III, a triplet of heavy left-handed fermions is added (various modifications and combinations of all these 3 mechanisms are also possible [25]).

All these mechanisms lead to non-conservation of the lepton number. We also note that the detailed experimental studies of Higgs boson properties, carried out after its discovery in 2012, are in good agreement with the predictions of the Standard Model: so far no signals of new physics have been detected in the Higgs sector [2].

We first consider the SSM of type I for a simple model of one lepton doublet $L_\mathrm{L} = (\nu_\mathrm{L}, e_\mathrm{L})^T$, interacting with a heavy right-handed neutrino (singlet) N_R.

The corresponding part of the full Lagrangian has the form

$$L_{\nu N} = \bar{N}_R i\gamma^\mu \partial_\mu N_R - \frac{1}{2} m_R \left(\bar{N}_R^c N_R + \bar{N}_R N_R^c\right) - y_\nu \left(\bar{L}_L \tilde{\varphi} N_R + \bar{N}_R \tilde{\varphi}^+ L_L\right), \tag{24}$$

where the mass is $m_R \gg v$ (it is assumed that it is generated by new physics not described by the SM). After spontaneous symmetry breaking on the v scale, the mass part of the Lagrangian arises, which is a superposition of the Dirac and Majorana mass terms

$$\mathcal{L}_{DM} = -m_D \left(\bar{\nu}_L N + \bar{N}\nu_L\right) - \frac{1}{2} m_R \bar{N} N, \tag{25}$$

where we introduced the notation for the Dirac mass

$$m_D = y_\nu \frac{v}{\sqrt{2}}, \tag{26}$$

and $N = N_R + N_R^c = N^c$ is a Majorana neutrino field. Next, we note that in view of $\nu_L^c \equiv (\nu_L)^c = (\nu^c)_R$ we have

$$\begin{gathered} \nu_L = \frac{1-\gamma^5}{2}\chi, \quad \chi = \nu_L + \nu_L^c = \chi^c, \\ \bar{\nu}_L N + \bar{N}\nu_L = \frac{1}{2}\left(\bar{\chi} N + \bar{N}\chi\right) + \frac{1}{2}\left(\bar{\chi}\gamma^5 N - \bar{N}\gamma^5\chi\right), \end{gathered} \tag{27}$$

and taking into account the relations (A.14), (A.16) and (A.17) (see Appendix A)

$$\begin{aligned} \bar{N}\gamma^5\chi &= -\chi^T (\gamma^5)^T \bar{N}^T = (-\chi^T C^{-1})(C(\gamma^5)^T C^{-1})(C\bar{N}^T) \\ &= \bar{\chi}^c \gamma^5 N^c = \bar{\chi}\gamma^5 N \end{aligned}$$

we represent (25) in the matrix form through the Majorana fields N and χ:

$$\mathcal{L}_{DM} = -\frac{1}{2}(\bar{\chi}, \bar{N}) \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \chi \\ N \end{pmatrix}. \tag{28}$$

After diagonalizing the mass matrix in (28),

$$\begin{gathered} U^T \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad U = RP, \\ R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad P = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}, \end{gathered} \tag{29}$$

we obtain its eigenvalues

$$m_{1,2} = \frac{1}{2} m_R \left(\sqrt{1 + 4\frac{m_D^2}{m_R^2}} \mp 1\right). \tag{30}$$

The matrix P is introduced in (29) in order to change the sign of the mass m_1 so that both masses m_1 and m_2 are positive. The corresponding eigenvectors of the mass matrix (Majorana fields with definite masses) are written as follows:

$$\begin{gathered} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = U^+ \begin{pmatrix} \chi \\ N \end{pmatrix}, \quad U^+ = \begin{pmatrix} i\cos\theta & -i\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \\ \nu_1 = i\chi\cos\theta - iN\sin\theta = -\nu_1^c, \\ \nu_2 = \chi\sin\theta + N\cos\theta = \nu_2^c, \\ \tan 2\theta = \frac{2m_D}{m_R}. \end{gathered} \tag{31}$$

As a result, (25) takes the form of the standard Majorana mass term (cf. (20))

$$\mathcal{L}_{DM} = -\frac{1}{2}\sum_{k=1}^{2} m_k \bar{\nu}_k \nu_k. \tag{32}$$

The initial flavor fields (included in the Lagrangian (24) and mass term (25)) turn out to be superpositions of Majorana fields with certain masses:

$$\begin{gathered} \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} = U \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}, \\ \nu_L = -i\nu_{1L}\cos\theta + \nu_{2L}\sin\theta, \\ N_R^c = i\nu_{1L}\sin\theta + \nu_{2L}\cos\theta. \end{gathered} \tag{33}$$

In the case of the heavy right-handed neutrino N_R we have $m_R \gg m_D$, and from (30) and (31) we obtain

$$m_1 \simeq \frac{m_D^2}{m_R} \ll m_D, \quad m_2 \simeq m_R, \quad \theta \simeq \frac{m_D}{m_R} \ll 1. \tag{34}$$

Therefore, if m_D is of the order of the mass of a charged fermion, then the mass of a light neutrino turns out to be very small even for the Yukawa coupling constant $y_\nu \sim 1$ due to the interaction with the heavy Majorana neutrino. This is the essence of the seesaw mechanism. In this case, as can be seen from (33) and (34), the active flavor neutrino ν_L contains a small admixture of the heavy Majorana neutrino. For example, for $m_D = m_t \simeq 173$ GeV ad $m_1 = 0.05$ eV we find $m_R \simeq m_t^2/m_1 \simeq 6\times 10^{14}$ GeV (see (14) and (23)), which is a typical grand unified scale [8–10].

In Appendix B, we consider a generalization of SSM corresponding to the modification of the mass matrix in (28):

$$\begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \to \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix},$$

where m_L is small but is not equal to zero.

2. EFFCTIVE LAGRANGIANS FOR SEESAW MECHANISM

2.1. We consider the effective Lagrangian $\mathcal{L}_{\rm eff}$ corresponding to the model (24) and suitable for energies

$E \ll m_{\rm R}$. It is determined by functional integration over heavy Majorana fields:

$$e^{iS_{\rm eff}} = \exp\left(i\int d^4x \mathcal{L}_{\rm eff}(x)\right) = \int [dN][d\bar{N}]\exp\left(i\int d^4x \mathcal{L}_{\nu N}(x)\right), \quad (35)$$

where the Lagrangian (24) is conveniently represented in the form

$$\begin{aligned}\mathcal{L}_{\nu N} &= \bar{N}\hat{K}N - \bar{N}J - \bar{J}N,\\ \hat{K} &= \frac{1}{2}(i\gamma^\mu\partial_\mu - m_{\rm R}),\\ J &= \frac{1}{2}y_\nu\left(\tilde{\varphi}^+ L_{\rm L} + \tilde{\varphi}^T L_{\rm L}^c\right),\\ \bar{J} &= \frac{1}{2}y_\nu\left(\bar{L}_{\rm L}\tilde{\varphi} + \bar{L}_{\rm L}^c\tilde{\varphi}^*\right).\end{aligned} \quad (36)$$

For deriving (36) we use the relations

$$\begin{aligned}N = N^c, \quad L_{\rm L}^c &= C\bar{L}_{\rm L}^T, \quad \bar{L}_L^c = -L^T C^{-1},\\ \bar{L}_L\tilde{\varphi}N &= -N^T\tilde{\varphi}^T C^{-1}C\bar{L}_{\rm L}^T = \bar{N}\tilde{\varphi}^T L_{\rm L}^c,\\ \bar{N}\tilde{\varphi}^+ L_L &= \bar{L}_{\rm L}^c\tilde{\varphi}^* N.\end{aligned} \quad (37)$$

The integration in (35) over fermion fields (taking into account methods of [26]) gives, in view of (36),

$$e^{iS_{\rm eff}} = \det\hat{K}\exp\left(-i\int d^4x \bar{J}\hat{K}^{-1}J\right). \quad (38)$$

The determinant $\det\hat{K}$ in (38) does not depend on the fields, and taking it into accounting adds only a constant term to the effective action, which can be omitted. As a result, we obtain the effective Lagrangian in the form

$$\mathcal{L}_{\rm eff} = -\bar{J}\hat{K}^{-1}J. \quad (39)$$

From here, putting $\hat{K}^{-1} \simeq -2/m_{\rm R}$ (in the leading order of expansion over $1/m_{\rm R}$), and taking into account (36), we obtain

$$\mathcal{L}_{\rm eff} = \frac{y_\nu^2}{2m_{\rm R}}\left(\bar{L}_{\rm L}\tilde{\varphi}\tilde{\varphi}^T L_{\rm L}^c + \bar{L}_{\rm L}^c\tilde{\varphi}^*\tilde{\varphi}^+ L_{\rm L}\right), \quad (40)$$

which is a special case of the operator (16). After substitution of (26), the effective Lagrangian (40) gives the corresponding mass term for the Majorana field χ introduced in (27):

$$\begin{aligned}\mathcal{L}_{\rm M} &= \frac{y_\nu^2 v^2}{4m_R}\left(\bar{\nu}_{\rm L}\nu_{\rm L}^c + \bar{\nu}_{\rm L}^c\nu_{\rm L}\right) = \frac{1}{2}m\bar{\chi}\chi,\\ m &= \frac{m_{\rm D}^2}{m_{\rm R}}, \quad \chi = \nu_{\rm L} + \nu_{\rm L}^c = \chi^c.\end{aligned} \quad (41)$$

The "incorrect" sign of the mass term (41) is removed by the transition to the Majorana field of negative charge parity χ':

$$\chi \to \chi' = i\chi = -\chi'^c. \quad (42)$$

Indeed, taking into account the relations (see Eq. (A.16) in Appendix A)

$$\bar{\chi}\chi = \bar{\chi}^c\chi = \chi^T C\chi = (-i\chi')^T C(-i\chi') = -\bar{\chi}'\chi', \quad (43)$$

we obtain

$$\mathcal{L}_{\rm M} = -\frac{1}{2}m\bar{\chi}'\chi'. \quad (44)$$

Formulas (41) and (44) agree with (31)–(34) in the leading order of the expansion of (39) over $1/m_{\rm R}$, as it should be.

2.2. Now we consider an extension of the SM with three lepton generations interacting with three heavy right-handed neutrinos $N_{j\rm R}$ $(j = 1,2,3)$. The corresponding interaction Lagrangian which generalizes (24), has the form [25, 27] (we use here and below the concise matrix notation):

$$\begin{aligned}\mathcal{L}_{\ell N} &= \bar{N}_{\rm R}i\gamma^\mu\partial_\mu N_{\rm R} - \frac{1}{2}\left(\bar{N}_{\rm R}^c M_{\rm R}N_{\rm R} + \bar{N}_{\rm R}M_{\rm R}^* N_{\rm R}^c\right)\\ &- \bar{L}_{\rm L}\tilde{\varphi}Y^\nu N_{\rm R} - \bar{N}_{\rm R}Y^{\nu+}\tilde{\varphi}^+ L_{\rm L},\end{aligned} \quad (45)$$

where $N_{\rm R} = (N_{j\rm R})$, the matrix of Yukawa coupling constants is denoted as $Y^\nu = (Y_{\alpha j}^\nu)$ and $M_{\rm R} = (M_{{\rm R}jk})$ is the complex symmetric 3×3 mass matrix: $M_{{\rm R}jk} = M_{{\rm R}kj}$ $(j,k = 1,2,3)$.

The matrix $M_{\rm R}$ is diagonalized by using the unitary matrix U:

$$\begin{aligned}U^T M_{\rm R}U &= M = {\rm diag}(M_1, M_2, M_3),\\ M_{\rm R} &= U^* M U^+,\end{aligned} \quad (46)$$

where M_j are real nonnegative numbers. This statement is a special case of Takagi's diagonalization theorem (see [28], Appendix D, and references therein). This theorem states that for any complex symmetric $n\times n$-matrix M_S there exists a unitary $n\times n$-matrix U such that

$$U^T M_S U = M = {\rm diag}(m_1, m_2, \ldots, m_n),$$

where all m_j are real and *nonnegative*. We prove this important theorem in Appendix C by modifying the proof from [28].

Applying (46), we represent the Lagrangian (45) in the mass basis of the heavy Majorana neutrinos:

$$\begin{aligned}\mathcal{L}_{\ell N} &= \frac{1}{2}\bar{N}_j\left(i\gamma\times\partial - M_j\right)N_j\\ &- \bar{L}_{\alpha L}\tilde{Y}_{\alpha j}^\nu\tilde{\varphi}N_j - \bar{N}_j\tilde{Y}_{j\alpha}^{\nu *}\tilde{\varphi}^+ L_{\alpha\rm L}.\end{aligned} \quad (47)$$

Here

$$\begin{aligned}N_j = \tilde{N}_{j\rm R} + \tilde{N}_{j\rm R}^c = N_j^c, \quad \tilde{N}_{\rm R} &= (\tilde{N}_{j\rm R}) = U^+ N_{\rm R},\\ \tilde{Y}^\nu &= Y^\nu U.\end{aligned} \quad (48)$$

Using obvious generalizations of the relations (37), we represent (47) in a form analogous to (36):

$$\begin{aligned} \mathcal{L}_{\ell N} &= \mathcal{L}_{\nu N} = \bar{N}\hat{K}N - \bar{N}J - \bar{J}N, \\ \hat{K} &= \frac{1}{2}(i\gamma\cdot\partial - M) = (\hat{K}_{jk}), \\ \hat{K}_{jk} &= \frac{1}{2}(i\gamma\cdot\partial - M_j)\delta_{jk}, \\ J &= \frac{1}{2}\left(\tilde{Y}^{\nu+}\tilde{\varphi}^{+}L_{\rm L} + \tilde{Y}^{\nu T}\tilde{\varphi}^{T}L_{\rm L}^{c}\right), \\ \bar{J} &= \frac{1}{2}\left(\bar{L}_{\rm L}\tilde{\varphi}\tilde{Y}^{\nu} + \bar{L}_{\rm L}^{c}\tilde{\varphi}^{*}\tilde{Y}^{\nu *}\right). \end{aligned} \tag{49}$$

Now we substitute (49) into (35) and use (38) and (39) to obtain an effective Lagrangian which generalizes (40):

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\bar{L}_{\rm L}\tilde{\varphi}\tilde{Y}^{\nu}M^{-1}\tilde{Y}^{\nu T}\tilde{\varphi}^{T}L_{\rm L}^{c} + {\rm H.c.} \tag{50}$$

Then, applying relations (46) and (48), we transform (50) to the form of the Weinberg operator (16):

$$\begin{aligned} \mathcal{L}_{\rm eff} &= z_{\alpha\beta}\bar{L}_{\alpha {\rm L}}\tilde{\varphi}\tilde{\varphi}^{T}L_{\beta {\rm L}}^{c} + {\rm H.c.}, \\ z_{\alpha\beta} &= \frac{1}{2}\left(Y^{\nu}M_{\rm R}^{-1}Y^{\nu T}\right)_{\alpha\beta}. \end{aligned} \tag{51}$$

After spontaneous symmetry breaking, the interaction with the Higgs doublet (51) generates the Majorana mass matrix of light neutrinos (cf. the case of one lepton generation (41)):

$$m_{\alpha\beta} = -z_{\alpha\beta}v^{2} = -(M_{\rm D}M_{\rm R}^{-1}M_{\rm D}^{T})_{\alpha\beta}, \quad M_{\rm D} = Y^{\nu}\frac{v}{\sqrt{2}}, \tag{52}$$

so that the mass term in the Lagrangian has the form

$$\mathcal{L}_{\rm M} = -\frac{1}{2}\left(m_{\alpha\beta}\bar{\nu}_{\alpha L}\nu_{\beta L}^{c} + {\rm H.c.}\right). \tag{53}$$

Diagonalization of the matrix (52) leads the mass term (53) to the standard form corresponding to three light Majorana neutrinos (see (20)).

2.3. The seesaw mechanism discussed above generates the Majorana neutrino mass. However, the nature of the neutrino mass (whether it is Majorana or Dirac) has not yet been established experimentally [2]. Therefore, it is interesting to consider this mechanism for Dirac neutrinos (see [29] and the literature cited there).

For a simple model with one lepton generation, the corresponding Yukawa part of the full Lagrangian has the form (cf. (24)):

$$\begin{aligned} \mathcal{L}_{\nu{\rm D}} &= -y_{\nu}\bar{L}_{\rm L}\tilde{\varphi}N_{\rm R} - m_{\rm R}\bar{N}_{\rm L}\nu_{\rm R} - m_{N}\bar{N}_{\rm L}N_{\rm R} - {\rm H.c.} \\ &= -y_{\nu}\left(\bar{L}_{\rm L}\tilde{\varphi}N + \bar{N}\tilde{\varphi}^{+}L_{\rm L}\right) \\ &\quad - m_{\rm R}\left(\bar{N}\nu_{\rm R} + \bar{\nu}_{\rm R}N\right) - m_{N}\bar{N}N, \end{aligned} \tag{54}$$

where the Dirac bispinor $N = N_{\rm L} + N_{\rm R}$ is a singlet with respect to the SM gauge group and describes "heavy" degrees of freedom under the assumption that the mass m_N is significantly larger than $m_{\rm R}$ and the typical electroweak scale v (see (5)).

The effective Lagrangian is obtained by substituting into (54) expressions for "heavy" fields via "light" fields (that is equivalent, in the accepted leading order of expansion over $1/m_N$, to the integration over heavy fermion fields in the generating functional, see [27, 30]):

$$\begin{aligned} N &= -\frac{1}{m_N}\left(y_{\nu}\tilde{\varphi}^{+}L_{\rm L} + m_{\rm R}\nu_{\rm R}\right), \\ \bar{N} &= -\frac{1}{m_N}\left(y_{\nu}\bar{L}_{\rm L}\tilde{\varphi} + m_{\rm R}\bar{\nu}_{\rm R}\right). \end{aligned}$$

These expressions follow from the equations of motion in the static approximation (neglecting the contribution of kinetic terms in the Lagrangian, which is justified in the energy region $E \ll m_N$):

$$\frac{\partial\mathcal{L}_{\nu{\rm D}}}{\partial\bar{N}} = 0, \quad \frac{\partial\mathcal{L}_{\nu{\rm D}}}{\partial N} = 0.$$

As a result, we obtain (cf. (40))

$$\mathcal{L}_{\rm eff} = \frac{y_{\nu}m_{\rm R}}{m_N}\left(\bar{L}_{\rm L}\tilde{\varphi}\nu_{\rm R} + \bar{\nu}_{\rm R}\tilde{\varphi}^{+}L_{\rm L}\right). \tag{55}$$

After spontaneous symmetry breaking (see (5)), from (55) the Dirac mass term follows (cf. (41)–(44))

$$\begin{aligned} \mathcal{L}_{\rm D} &= m_{\nu}\left(\bar{\nu}_{\rm L}\nu_{\rm R} + \bar{\nu}_{\rm R}\nu_{\rm L}\right) = m_{\nu}\bar{\nu}\nu = -m_{\nu}\bar{\nu}'\nu', \\ \nu' &= \gamma^{5}\nu = \gamma^{5}\left(\nu_{\rm L} + \nu_{\rm R}\right) = \nu_{\rm R} - \nu_{\rm L}, \\ m_{\nu} &= \frac{m_{\rm L}m_{\rm R}}{m_N}, \quad m_{\rm L} = y_{\nu}\frac{v}{\sqrt{2}}, \\ m_{\nu} &\ll m_{\rm L,R} \ll m_N. \end{aligned} \tag{56}$$

Here the correct sign of the mass is provided by making use of the γ^5-transformation of the Dirac bispinor [31]:

$$\bar{\nu}\nu = \nu^{+}\gamma^{0}\nu = \nu'^{+}\gamma^{5}\gamma^{0}\gamma^{5}\nu' = -\bar{\nu}'\left(\gamma^{5}\right)^{2}\nu' = -\bar{\nu}'\nu'.$$

There are three possible relationships for the mass parameters:

$$1)\, m_{\rm L} \sim m_{\rm R}, \quad 2)\, m_{\rm L} \ll m_{\rm R}, \quad 3)\, m_{\rm L} \gg m_{\rm R}.$$

As it was shown in [29], the case 3), called the undemocratic Dirac seesaw mechanism (with an appropriate generalization to several lepton generations), can be used to describe baryon asymmetry of the Universe, as well as the stability of the dark matter.

2.4. Following the work [32], we consider a generalization of the model (54) to three lepton generations (for another generalization, see [29]). This generalization is based on the extended symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y \times Z_4 \times Z_2$. The group Z_4 (which describes a discrete analogue of the lepton number) prohibits the Majorana terms in the Lagrangian and provides the stability of candidates for dark matter particles, while the group Z_2 provides a seesaw mechanism for the generation of Dirac neutrino mass, prohibiting tree amplitudes that connect left-handed and right-handed neutrinos (for discrete groups Z_n see [33]). In addition to standard fermions, the model includes three heavy Dirac fermions N_i and three scalars (in addition to the Higgs doublet) χ, ζ, η, which are gauge singlets. The scalar χ is uncharged with respect to the group Z_4 but is odd with respect to Z_2, the other two scalars are uncharged in Z_2.

The part of the Lagrangian of the model responsible for the generation of the mass of light (active) neutrinos has the form

$$\mathcal{L}_{\nu N\chi} = -f_{\alpha i}\bar{L}_{\alpha \mathrm{L}}\tilde{\varphi}N_{i\mathrm{R}} - g_{i\alpha}\bar{N}_{i\mathrm{L}}\chi\nu_{\alpha\mathrm{R}} - M_{ij}\bar{N}_{i\mathrm{L}}N_{j\mathrm{R}} - \mathrm{H.c.}, \quad (57)$$

where $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$. The corresponding effective Lagrangian is obtained by means of obvious generalization of the method outlined in Section 2.3 (we use the index free matrix notation):

$$\mathcal{L}_{\mathrm{eff}} = \bar{L}_{\mathrm{L}} f\tilde{\varphi}M^{-1}g\chi\nu_{\mathrm{R}} + \mathrm{H.c.} \quad (58)$$

After spontaneous symmetry breaking, the scalars $\tilde{\varphi}$ and χ obtain vacuum expectation values (see (5)):

$$\langle 0|\tilde{\varphi}|0\rangle = (v/\sqrt{2}, 0)^T, \quad \langle 0|\chi|0\rangle = u, \quad (59)$$

and as a result, the Dirac mass term is generated

$$\mathcal{L}_{\mathrm{D}} = \bar{\nu}_{\mathrm{L}}M_\nu\nu_{\mathrm{R}} + \mathrm{H.c.},$$

where the mass matrix is (cf. (56))

$$M_\nu = \frac{vu}{\sqrt{2}} fM^{-1}g. \quad (60)$$

Note that the Z_4-charged scalars ζ and η (in contrast to the neutral φ and χ) have zero vacuum expectation values, so that the group Z_4 remains unbroken after spontaneous breaking of electroweak symmetry, while the vacuum expectation value of the scalar χ (see (59)), which is odd with respect to Z_2, breaks Z_2-symmetry spontaneously, which generates small Dirac masses (see (60) for $|M_{ij}| \gg v, u$).

Besides, the analysis shows [32] that particles ζ turn out to be stable, and therefore can be considered as candidates for the dark matter particles, while η particles are unstable.

3. CONCLUSIONS

We have considered the Majorana and Dirac versions of the seesaw mechanism (of type I) for generating the mass of active light neutrinos, which is based on the extension of the SM by adding heavy neutrinos that have a Yukawa interaction with standard flavor neutrinos. By integrating over the "heavy" fields in the generating functional of the theory, the corresponding low-energy effective Lagrangians were obtained, which, after spontaneous breaking of electroweak symmetry, lead to the mass terms of light neutrinos.

We stress the fundamental difference between the mechanisms of mass generation of charged leptons (and quarks) and light neutrinos: the masses of the former are determined by the products of the electroweak scale v and the corresponding *dimensionless* Yukawa coupling constants (see (10)), while the smallness of the masses of the second (active neutrinos) is ensured by the introduction of the *dimensional* parameter the large mass scale of new physics Λ, so that the mass value is suppressed by the small ratio v/Λ (see (23)).

In the case of the considered seesaw mechanism (SSM), the scale Λ represents the scale of the masses M of heavy neutrinos introduced during the extension of the Standard Model. However, a natural question arises (see, for example, [34]) about the generation of the scale M itself, which was introduced above "by hands" (see m_{R} in (24) and m_N in (54)). As it can be seen from (23), the scale of Λ coincides in order of magnitude with the typical scale of Grand Unified Theories (GUT). An example of an extension of the SM leading to SSM is GUT, based on the gauge group SO(10) [20]. There are many ways of spontaneous breaking of this group up to the SM group $\mathrm{G_{SM}} = \mathrm{SU(3)}_c \times \mathrm{SU(2)_L} \times \mathrm{U(1)}_Y$ with its subsequent breaking on the scale v, besides the minimal way contains two steps (see [35], where the supersymmetric SO(10) theory is considered):

$$\mathrm{SO(10)} \xrightarrow{\Lambda_{\mathrm{GUT}}} \mathrm{G_{SM}} \xrightarrow{v} \mathrm{SU(3)}_c \times \mathrm{U(1)_{em}}.$$

Non-supersymmetric SO(10)-GUT is considered in [36], where, in particular, it is shown how, in the case of spontaneous breaking of SO(10), the SSM of type I arises for light neutrinos.

Note that left-right symmetric theories also lead to the SSM [23, 24].

The Majorana SSM leads to non-conservation of the lepton number L, changing it by two units (see (50) and (53)). This opens up the possibility of observing numerous physical processes with $|\Delta L| = 2$ induced by Majorana neutrinos: neutrinoless double beta decay of nuclei (see [37]) and its analogs—semileptonic decays of mesons with the birth of a pair of leptons with identical electric charges (dileptons) [38, 39], production of dileptons in deep inelastic proton-proton and lepton-proton collisions at high-energy colliders (see, for example, [38, 40]), etc. The search for such processes is one of the important areas of researches in particle physics [2].

Heavy Majorana neutrinos can play a significant role in cosmology: their decays with CP violation at the early stages of the Universe evolution lead to lepton asymmetry that, due to the special non-perturbative electroweak interaction of leptons and quarks with non-conservation of lepton and baryon numbers, is transformed into baryon asymmetry. This mechanism for generating baryon asymmetry in the Universe is called the leptogenesis (see [41] and the references therein). An extension of the Standard Model by adding three heavy right-handed neutrinos is called the neutrino minimal Standard Mode (νMSM). Its applications in cosmology, including the problems of baryogenesis and dark matter, are considered, for example, in [42, 43].

In what concerns the Dirac SSM, as it was indicated above, by considering the corresponding extension of the SM, new scalar singlets are introduced, and one of these singlets can serve as a candidate for the role of particles of the stable dark matter.

Thus, the small masses of active neutrinos serve as a clear signal of new physics, which is not described by the Standard Model [34].

APPENDIX A

WEYL, DIRAC AND MAJORANA SPINORS IN MINKOWSKI SPACE $\mathbb{R}^{1,3}$

To describe fermion fields with spin 1/2, two-component complex Weyl spinors are used, which are transformed independently in the fundamental (spinor ξ_a, $a = 1,2$) and antifundamental, or complex conjugate to the fundamental, (spinor $\eta^{\dot{a}}$, $\dot{a} = \dot{1},\dot{2}$) representations of the group $\mathrm{SL}(2,\mathbb{C})$(here we follow [28, 44, 45]):

$$\xi'_a = A_a^b \xi_b, \quad \eta'^{\dot{a}} = \tilde{A}^{\dot{a}}_{\dot{b}} \eta^{\dot{b}}; \\ \tilde{A} = (A^{-1})^+, \quad \det A = 1. \tag{A.1}$$

As it is known, $\mathrm{SL}(2,\mathbb{C})$ is a double covering group for the proper orthochronous Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$. The complex 2×2-matrix $A \in \mathrm{SL}(2,\mathbb{C})$ corresponds to a real pseudo-orthogonal 4×4-matrix $\Lambda \in \mathrm{SO}^{\uparrow}(1,3)$:

$$\Lambda_{\mu\nu} = g_{\mu\lambda}\Lambda^{\lambda}_{\nu} = \frac{1}{2}\operatorname{tr}\left(\sigma_\mu A \tilde{\sigma}_\nu A^+\right), \\ \mu, \nu, \ldots = 0,1,2,3, \tag{A.2}$$

where $g_{\mu\lambda} = \mathrm{diag}(+1,-1,-1,-1)$ and two sets of four 2×2-matrices were introduced, including Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and unit matrix $\sigma_0 = I$:

$$\tilde{\sigma}^\mu = \left\|(\tilde{\sigma}^\mu)^{\dot{c}a}\right\| = (I, -\boldsymbol{\sigma}), \quad \sigma^\mu = \left\|(\sigma^\mu)_{a\dot{c}}\right\| = (I, \boldsymbol{\sigma}), \tag{A.3}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

Note that the double covering means that two elements of the group $\mathrm{SL}(2,\mathbb{C})$ correspond, as follows from (A.2), to one element of the Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$: $\pm A \to \Lambda$. That is why, the spinor representations of the Lorentz group are called double-valued and the physical observables may not be the spinor fermion fields themselves but their bilinear combinations.

The Dirac 4-component spinor (bispinor) ψ is composed of two Weyl spinors (A.1):

$$\psi = \begin{pmatrix} \xi_{\mathrm{L}} \\ \eta_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \xi_a \\ \eta^{\dot{a}} \end{pmatrix}, \quad \xi_{\mathrm{L}} := (\xi_a) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \\ \eta_{\mathrm{R}} := (\eta^{\dot{a}}) = \begin{pmatrix} \eta^{\dot{1}} \\ \eta^{\dot{2}} \end{pmatrix}, \tag{A.5}$$

which corresponds to the Weyl representation of gamma matrices (see (A.3)):

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (I, \boldsymbol{\sigma}), \quad \tilde{\sigma}^\mu = (I, -\boldsymbol{\sigma}), \\ \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = (\gamma^k) = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \\ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{A.6}$$

The Dirac conjugated bispinor to the bispinor (A.5), has the form[2]

$$\bar{\psi} = \psi^+\gamma^0 = \left(\bar{\xi}_{\dot{a}}, \bar{\eta}^a\right)\gamma^0 = \left(\bar{\eta}^a, \bar{\xi}_{\dot{a}}\right), \tag{A.7}$$

$$\bar{\xi}_{\dot{a}} = (\xi_a)^*, \quad \bar{\eta}^a = \left(\eta^{\dot{a}}\right)^*. \tag{A.8}$$

The left-handed ψ_{L} and the right-handed ψ_{R} bispinors, which compose the bispinor (A.5), are expressed in terms of Weyl spinors as follows:

$$\psi_{\mathrm{L}} = \frac{1}{2}(1 - \gamma^5)\psi = \begin{pmatrix} \xi_a \\ 0 \end{pmatrix}, \\ \psi_{\mathrm{R}} = \frac{1}{2}(1 + \gamma^5)\psi = \begin{pmatrix} 0 \\ \eta^{\dot{a}} \end{pmatrix}, \\ \psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}, \tag{A.9}$$

where the matrix γ^5 is defined in (A.6).

The covering group $\mathrm{SL}(2,\mathbb{C})$ of the Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$ acts on the bispinor (A.5) according to (A.1):

$$\psi(x) \to \psi'(x') = \begin{pmatrix} A & 0 \\ 0 & \tilde{A} \end{pmatrix}\psi(x) \equiv D(A)\psi(x), \\ D(A) = \exp\left(\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}\right), \\ \Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}, \tag{A.10}$$

$$(\sigma^{\mu\nu})_a^{\ b} = (\sigma^\mu)_{a\dot{c}}(\tilde{\sigma}^\nu)^{\dot{c}b} - (\mu \leftrightarrow \nu), \\ (\tilde{\sigma}^{\mu\nu})^{\dot{a}}_{\ \dot{b}} = (\tilde{\sigma}^\nu)^{\dot{a}c}(\sigma^\mu)_{c\dot{b}} - (\mu \leftrightarrow \nu),$$

[2] The Dirac conjugated bispinor $\bar{\psi}$ is constructed in a such way that it can be covariantly contracted with the spinor ψ. In the definition of $\bar{\psi}$, the matrix γ^0 only formally coincides with the Dirac matrix γ^0, as it can be seen from the arrangement of its dotted and undotted spinor indices; see (A.3).

where the matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is composed of 6 real parameters of the Lorentz group. This follows from the fact that any complex 2×2-matrix $A \in \mathrm{SL}(2,\mathbb{C})$ is represented as a product of Hermitian and unitary matrices which respectively correspond to a real pseudo-orthogonal 4×4 boost matrix $\Lambda \in \mathrm{SO}^{\uparrow}(1,3)$ and a pure three-dimensional rotation matrix in $\mathbb{R}^{1,3}$. Taking into account the Lorentz transformation for the coordinates $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$, the transformation (A.10) can be represented as

$$\psi'(x) = \exp\left(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\psi(x), \quad J^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) + \frac{1}{2}\Sigma^{\mu\nu}, \tag{A.11}$$

where $J^{\mu\nu}$ are complete generators of the Lorentz group, including generators $\Sigma^{\mu\nu}$ in the spinor representation.

Using the complex conjugation (A.8) one can construct for the bispinor (A.5) another conjugated bispinor

$$\psi^c = \begin{pmatrix} 0 & \varepsilon \\ \tilde{\varepsilon} & 0 \end{pmatrix}\psi^* = \begin{pmatrix} \varepsilon_{ab}\bar{\eta}^b \\ \varepsilon^{\dot{a}\dot{c}}\bar{\xi}_{\dot{c}} \end{pmatrix} = \begin{pmatrix} \bar{\eta}_a \\ \bar{\xi}^{\dot{a}} \end{pmatrix}, \tag{A.12}$$

which is transformed according to the same law (A.10) as the initial bispinor ψ given in (A.5). Here we introduce matrices

$$\varepsilon = \|\varepsilon_{ab}\|, \quad \tilde{\varepsilon} = \|\varepsilon^{\dot{a}\dot{c}}\|, \quad \varepsilon = -\tilde{\varepsilon} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.13}$$

that lower $\bar{\eta}_a = \varepsilon_{ab}\bar{\eta}^b$ and raise $\bar{\xi}^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\bar{\xi}_{\dot{b}}$ indices of the components of Weyl spinors. The bispinor ψ^c, defined in (A.12), is called the *charge conjugated to the bispinor* ψ (65). The definition of charge conjugation (A.5) is written in matrix form as follows

$$\psi^c = C\bar{\psi}^T = C(\gamma^0)^T\psi^*, \quad C = \begin{pmatrix} \varepsilon & 0 \\ 0 & \tilde{\varepsilon} \end{pmatrix} = i\gamma^2\gamma^0, \tag{A.14}$$

and the charge conjugation is an involutive operation $(\psi^c)^c = \psi$. The charge conjugation matrix C satisfies the relations

$$(\gamma^{\mu})^T = -C^{-1}\gamma^{\mu}C, \; C^T = -C, \quad C^+ = C^{-1}, \; C^{-1} = -C. \tag{A.15}$$

The last two relations in (A.15) are valid only in specific representations of the Dirac matrices, for example, in the Weyl representation used. From (A.14) and (A.15) we find the Dirac conjugated bispinor to ψ^c

$$\overline{\psi^c} \equiv \bar{\psi}^c = \psi^T C = -\psi^T C^{-1}. \tag{A.16}$$

Note that the first relation in (A.15) is actually the definition of the matrix C, since it precisely guarantees that if the bispinor ψ describes a particle with charge e, then the bispinor ψ^c describes a particle with charge $-e$, i.e. an antiparticle (see, for example, [46]). From this relation it also follows that

$$(\gamma^5)^T = C^{-1}\gamma^5 C. \tag{A.17}$$

Taking into account (A.17) and $(\gamma^{\mu})^+ = \gamma^0\gamma^{\mu}\gamma^0$, we obtain

$$(\gamma^5)^* = -C^{-1}\gamma^0\gamma^5\gamma^0 C,$$

whence for the left-handed and right-handed components of the bispinor ψ^c we have:

$$\psi^c_{\mathrm{L}} \equiv (\psi_{\mathrm{L}})^c = \frac{1}{2}(1+\gamma^5)\psi^c = (\psi^c)_{\mathrm{R}}, \quad \psi^c_{\mathrm{R}} \equiv (\psi_{\mathrm{R}})^c = \frac{1}{2}(1-\gamma^5)\psi^c = (\psi^c)_{\mathrm{L}}. \tag{A.18}$$

Since the bispinors ψ and ψ^c, as it follows from the comparison of (A.5) and (A.12), are equally transformed with respect to $\mathrm{SL}(2,\mathbb{C})$, they can be equated:

$$\psi = \psi^c. \tag{A.19}$$

A bispinor ψ equal to its charge-conjugated bispinor ψ^c is called a *Majorana bispinor*. Note that the definition (A.19) can be generalized by including an arbitrary phase factor [45, 47]: $\psi^c = e^{i\theta}\psi$, which is sometimes convenient (see, for example, (31)), but one can always choose $\theta = 0$ by accordingly redefining the fermion field ψ.

The condition (A.19) is not invariant under $U(1)$-transformations $\psi \to e^{i\alpha}\psi$, and this means that Majorana fermions are truly neutral particles that are identical to their antiparticles, i.e. cannot have conserved additive quantum numbers associated with $U(1)$-symmetries: electric charge and any fermion numbers (lepton, baryon, etc.).

The condition (A.19) is equivalent (see (A.5) and (A.12)) to the equality of the Weyl spinors, that make up the Dirac bispinor: $\xi_a = \bar{\eta}_a$, and the Majorana bispinor is represented in the form

$$\psi_{\mathrm{M}} = \begin{pmatrix} \xi_a \\ \bar{\xi}^{\dot{a}} \end{pmatrix} \equiv \begin{pmatrix} \bar{\eta}_a \\ \eta^{\dot{a}} \end{pmatrix}, \quad \bar{\xi}^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}}(\xi_b)^*, \quad \bar{\eta}_a = \varepsilon_{ab}(\eta^{\dot{b}})^*, \tag{A.20}$$

i.e., it is determined by only one left-handed ξ_a, or right-handed $\eta^{\dot{a}}$, Weyl spinor.

From (A.14), (A.16), (A.19) and (A.15) we deduce

$$\bar{\psi}_{\mathrm{M}} = -\psi^T_{\mathrm{M}}C^{-1} = \psi^T_{\mathrm{M}}C = (\xi^a, \bar{\xi}_{\dot{a}}), \tag{A.21}$$

which allows us to write the Majorana mass term in the Lagrangian as

$$-\mathcal{L}_{\rm M} = \frac{m}{2}\overline{\psi}_{\rm M}\psi_{\rm M} = \frac{m}{2}\psi_{\rm M}^T C\psi_{\rm M} = \frac{m}{2}\left(\xi^a\xi_a + \overline{\xi}_{\dot a}\overline{\xi}^{\dot a}\right) = \frac{m}{2}\left(\xi^a\varepsilon_{ab}\xi^b + \overline{\xi}_{\dot a}\varepsilon^{\dot a\dot b}\overline{\xi}_{\dot b}\right), \quad (A.22)$$

where the factor 1/2 is introduced so that the coefficient 2 does not appear for mass m in the Dirac equation obtained by varying in ξ and $\overline{\xi}$. Using (A.20), Eq. (A.22) can be rewritten in the equivalent form by making the substitutions $\xi \to \overline{\eta}$ and $\overline{\xi} \to \eta$. We emphasize that the convolutions of the quadratic combinations of the components ξ and $\overline{\xi}$ with antisymmetric ε-symbols in (A.22) are nonzero due to the anticommutativity of the components of the fermionic fields ξ and $\overline{\xi}$. This anticommutativity also ensures that the electromagnetic current for Majorana fermions is equal to zero:

$$\overline{\psi}_{\rm M}\gamma^\mu\psi_{\rm M} = \overline{\psi}^c_{\rm M}\gamma^\mu\psi^c_{\rm M} = -\psi^T_{\rm M}C^{-1}\gamma^\mu C\overline{\psi}^T_{\rm M} = \overline{\psi}_{\rm M}C(\gamma^\mu)^T C^{-1}\psi_{\rm M} = -\overline{\psi}_{\rm M}\gamma^\mu\psi_{\rm M} = 0,$$

where relations (A.19), (A.16) and (A.15) are taken into account.

Note that the kinetic term for the Majorana field is written in the form

$$\frac{1}{2}\overline{\psi}_{\rm M}i\gamma^\mu\partial_\mu\psi_{\rm M} = \frac{i}{2}\left(\xi^a(\sigma^\mu)_{a\dot b}\partial_\mu\overline{\xi}^{\dot b} + \overline{\xi}_{\dot a}(\tilde\sigma^\mu)^{\dot a b}\partial_\mu\xi_b\right). \quad (A.23)$$

According to (A.22) and (A.23), the free Majorana field Lagrangian has the form

$$\mathcal{L}_{\rm f} = \frac{1}{2}\overline{\psi}_{\rm M}\left(i\gamma^\mu\partial_\mu - m\right)\psi_{\rm M} = \frac{1}{2}\xi^{c+}_{\rm L}\left(i\sigma^\mu\partial_\mu\xi^c_{\rm L} - m\xi_{\rm L}\right) + \frac{1}{2}\xi^+_{\rm L}\left(i\tilde\sigma^\mu\partial_\mu\xi_{\rm L} - m\xi^c_{\rm L}\right), \quad (A.24)$$

where the compact matrix notation is used (see (A.6), (A.20) and (A.21)):

$$\psi_{\rm M} = \begin{pmatrix}\xi_{\rm L}\\ \xi^c_{\rm L}\end{pmatrix},\quad \overline{\psi}_{\rm M} = \left(\xi^{c+}_{\rm L}, \xi^+_{\rm L}\right),\quad \xi_L = (\xi_a),\quad \xi^c_L = i\sigma_2\xi^*_{\rm L} = (\varepsilon^{\dot a\dot b}\overline{\xi}_{\dot b}). \quad (A.25)$$

From (A.24) the equation of motion in bispinor form follows

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi_{\rm M} = 0,$$

as well as the equivalent pair of equations for Weyl spinors

$$i\tilde\sigma^\mu\partial_\mu\xi_{\rm L} - m\xi^c_{\rm L} = 0,\quad i\sigma^\mu\partial_\mu\xi^c_{\rm L} - m\xi_{\rm L} = 0, \quad (A.26)$$

and the second equation is obtained by complex conjugation from the first one. For a massless fermion with a given 4-momentum $p^\mu = (|\mathbf{p}|, \mathbf{p})$ the bispinor $\psi_{\rm M}(x) \sim \exp(-ip\cdot x)$ and, as follows from (A.26) in view of (A.3), the spinors $\xi_L(x)$ and $\xi^c_L(x)$ obey independent equations

$$\boldsymbol{\sigma}\cdot\mathbf{n}\xi_{\rm L} = -\xi_{\rm L},\quad \boldsymbol{\sigma}\cdot\mathbf{n}\xi^c_{\rm L} = \xi^c_{\rm L},\quad \mathbf{n} = \mathbf{p}/|\mathbf{p}|,$$

that is, these spinors are indeed left-handed and right-handed, respectively (see (A.18)).

Now we compare the Majorana mass term (82) with the Dirac one:

$$-\mathcal{L}_{\rm D} = m_{\rm D}\overline{\psi}\psi = m_{\rm D}\left(\eta^+_{\rm R}\xi_{\rm L} + \xi^+_{\rm L}\eta_{\rm R}\right) = m_{\rm D}\left(\overline{\eta}^a\xi_a + \overline{\xi}_{\dot a}\eta^{\dot a}\right), \quad (A.27)$$

where we make use of the notation (see (A.5) and (A.7))

$$\psi = \begin{pmatrix}\xi_{\rm L}\\ \eta_{\rm R}\end{pmatrix},\quad \xi_{\rm L} = (\xi_a),\quad \eta_{\rm R} = (\eta^{\dot a}). \quad (A.28)$$

Thus, the Dirac mass term arises only in the presence of both left-handed $\xi_{\rm L}$ and right-handed η_R Weyl spinors, which are *independent* components of the Dirac bispinor (in contrast to the Majorana bispinor, for which the mass term is determined either by only left-handed $\xi_{\rm L}$, or by only right-handed $\eta_{\rm R} = \xi^c_{\rm L}$, Weyl components).

APPENDIX B

MASS TERM OF GENERAL FORM

Now we consider the general mass term including the Majorana terms of types L and R and the Dirac term:

$$-\mathcal{L}_{\rm MD} = \frac{1}{2}m_{\rm L}\left(\overline{\psi}_{\rm L}\psi^c_{\rm L} + \overline{\psi}^c_{\rm L}\psi_{\rm L}\right) + \frac{1}{2}m_{\rm R}\left(\overline{\psi}_{\rm R}\psi^c_{\rm R} + \overline{\psi}^c_{\rm R}\psi_{\rm R}\right) + m_{\rm D}\left(\overline{\psi}_{\rm L}\psi_{\rm R} + \overline{\psi}_{\rm R}\psi_{\rm L}\right). \quad (B.1)$$

Let us introduce, following [31], Majorana bispinors

$$\lambda = \psi_{\rm L} + \psi^c_{\rm L} = \lambda^c,\quad \rho = \psi_{\rm R} + \psi^c_{\rm R} = \rho^c,\quad \lambda = \begin{pmatrix}\xi_{\rm L}\\ \xi^c_{\rm L}\end{pmatrix},\quad \rho = \begin{pmatrix}\eta^c_{\rm R}\\ \eta_{\rm R}\end{pmatrix} \quad (B.2)$$

and, taking into account (A.18), we express in their terms the bispinors included in (B.1):

$$\psi_{\rm L} = \frac{1}{2}(1-\gamma^5)\lambda,\quad \psi^c_{\rm L} = \frac{1}{2}(1+\gamma^5)\lambda,\quad \psi_{\rm R} = \frac{1}{2}(1+\gamma^5)\rho,\quad \psi^c_{\rm R} = \frac{1}{2}(1-\gamma^5)\rho. \quad (B.3)$$

Substituting (B.3) into (B.1), we obtain a more convenient representation of the total mass term:

$$-\mathcal{L}_{\mathrm{MD}} = \frac{1}{2}m_{\mathrm{L}}\bar{\lambda}\lambda + \frac{1}{2}m_{\mathrm{R}}\bar{\rho}\rho + \frac{1}{2}m_{\mathrm{D}}\left(\bar{\lambda}\rho + \bar{\rho}\lambda\right) = \frac{1}{2}(\bar{\lambda},\bar{\rho})\begin{pmatrix} m_{\mathrm{L}} & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_{\mathrm{R}} \end{pmatrix}\begin{pmatrix} \lambda \\ \rho \end{pmatrix}. \quad \text{(B.4)}$$

Diagonalizing the symmetric mass matrix in (B.4) by using the unitary matrix U (see Eq. (29) above in the main text), we deduce

$$U^T\begin{pmatrix} m_{\mathrm{L}} & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_{\mathrm{R}} \end{pmatrix}U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \text{(B.5)}$$

where

$$U = \begin{pmatrix} -i\cos\theta & \sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \quad \tan 2\theta = \frac{2m_{\mathrm{D}}}{m_{\mathrm{R}} - m_{\mathrm{L}}}, \quad \text{(B.6)}$$
$$m_{1,2} = \frac{1}{2}\left[\sqrt{(m_{\mathrm{R}} - m_{\mathrm{L}})^2 + 4m_{\mathrm{D}}^2} \mp (m_{\mathrm{R}} + m_{\mathrm{L}})\right].$$

As a result, (B.4) takes the form of a mass term for two Majorana fermions

$$-\mathcal{L}_{\mathrm{MD}} = \frac{1}{2}m_1\bar{\chi}_1\chi_1 + \frac{1}{2}m_2\bar{\chi}_2\chi_2,$$
$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = U^+\begin{pmatrix} \lambda \\ \rho \end{pmatrix}, \quad \text{(B.7)}$$
$$\chi_1 = i\lambda\cos\theta - i\rho\sin\theta,$$
$$\chi_2 = \lambda\sin\theta + \rho\cos\theta,$$

where the Majorana bispinors λ and ρ were defined in (B.2).

Thus, the general mass term (B.1) for the bispinor (A.28) composed of two *independent* Weyl spinors is in fact a mass term for two Majorana fermions with different masses.

Let there be the following hierarchy of mass parameters in (B.1):

$$m_{\mathrm{L}} \ll m_{\mathrm{D}} \ll m_{\mathrm{R}}. \quad \text{(B.8)}$$

Then from (B.6) and (B.7) we find

$$\theta \simeq \frac{m_{\mathrm{D}}}{m_{\mathrm{R}}} \ll 1,$$
$$\chi_1 \simeq i\lambda - i\theta\rho, \; \chi_2 \simeq \theta\lambda + \rho, \quad \text{(B.9)}$$
$$m_1 \simeq \frac{m_{\mathrm{D}}^2}{m_{\mathrm{R}}} - m_{\mathrm{L}}, \quad m_2 \simeq \frac{m_{\mathrm{D}}^2}{m_{\mathrm{R}}} + m_{\mathrm{R}}.$$

It follows that under the conditions (B.8), the fermion χ_2 turns out to be heavy ($m_2 \gg m_1$), while the fermion χ_1 is light. In the case of $m_{\mathrm{L}} = 0$, we arrive at the *seesaw mechanism* (SSM) of the generation of a small mass due to a large mass,

$$m_1 \simeq \frac{m_{\mathrm{D}}^2}{m_{\mathrm{R}}} \ll m_R, \quad m_2 \simeq m_{\mathrm{R}};$$
$$\chi_1 \simeq i\lambda = i(\psi_{\mathrm{L}} + \psi_{\mathrm{L}}^c), \quad \chi_2 \simeq \rho = \psi_{\mathrm{R}} + \psi_{\mathrm{R}}^c.$$

This mechanism was discussed in the main text (see Section 1) with relation to neutrino physics.

Note also that in the standard SSM the choice of $m_{\mathrm{L}} = 0$ is made due to the fact that for the generation of $m_{\mathrm{L}} \neq 0$ it is required to introduce the Higgs triplet into the theory: the L-type Majorana mass term in (B.1) for neutrinos carries weak isospin 1.

When fixing conditions

$$m_{\mathrm{L}} = m_{\mathrm{R}} = 0, \quad \text{(B.10)}$$

we obtain the usual Dirac fermion, which corresponds, as follows from (B.6) and (B.7), to the degenerate case of two Majorana fermions [31]:

$$m_1 = m_2 = m_{\mathrm{D}},$$
$$\chi_1 = \frac{i}{\sqrt{2}}(\lambda - \rho), \quad \chi_2 = \frac{1}{\sqrt{2}}(\lambda + \rho). \quad \text{(B.11)}$$

Taking into account (B.3), from (B.2) we find a representation of the Dirac field in the form of a superposition of two Majorana fields with the same mass (see also [31, 48])

$$\psi = \frac{1}{\sqrt{2}}(\chi_2 + i\gamma^5\chi_1).$$

A small deviation from the case of Dirac fermions leads to quasi-Dirac fermions (*quasi-Dirac*, or *pseudo-Dirac, neutrinos* in neutrino physics, see [49]). Having determined two small parameters

$$\epsilon = \frac{m_{\mathrm{R}} + m_{\mathrm{L}}}{2m_{\mathrm{D}}} \ll 1, \quad \delta = \frac{m_{\mathrm{R}} - m_{\mathrm{L}}}{4m_{\mathrm{D}}} \ll 1,$$

and taking into account (B.6) and (B.7), we obtain (cf. (B.11)))

$$\chi_1 = \frac{i}{\sqrt{2}}[(1+\delta)\lambda - (1-\delta)\rho)], \quad m_1 = m_{\mathrm{D}}(1-\epsilon),$$
$$\chi_2 = \frac{1}{\sqrt{2}}[(1-\delta)\lambda + (1+\delta)\rho)], \quad m_2 = m_{\mathrm{D}}(1+\epsilon).$$

APPENDIX C

DIAGONALIZATION OF THE MASS MATRIX FOR MAJORANA FERMIONS

In the Lagrangians describing the seesaw mechanism (for the case of several generations of neutrinos), the Majorana mass terms are written (after spontaneous symmetry breaking) with the help of the complex symmetric mass matrix M_{R} (see (45)). Let the number of the generations of neutrinos be equal to n. To pass to the neutrino states with definite masses, the symmetric complex $n \times n$-matrix M_R has to be diagonalized by using the unitary transformation of right-handed Majorana neutrinos $N_R \to U N_R$, where $U \in U(n)$. This leads to the transformation of the mass matrix $M_R \to U^T M_R U$. The following statement known in the literature as *Takagi's diagonalization theorem* holds (see Appendix D in [28] as well as the references therein).

Theorem. *For any complex symmetric $n \times n$-matrix $M_{\mathbb{C}}$ there exists a unitary matrix U such that*

$$U^T M_{\mathbb{C}} U = \mathrm{diag}(m_1, m_2, \ldots, m_n) \equiv M_D, \qquad (C.1)$$

where all parameters m_j are real and non-negative.

Proof. The proof is based on the explicit construction of the unitary matrix U, which diagonalizes $M_{\mathbb{C}}$ according to (C.1). The complex symmetric $n \times n$-matrix $M_{\mathbb{C}}$ is representable in the form $M_{\mathbb{C}} = X + iY$, where X, Y are real symmetric $n \times n$ matrices. Introduce a symmetric real $2n \times 2n$-matrix that is the 2 × 2-block matrix with $n \times n$-blocks X, Y:

$$M = \begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix} = \sigma_3 \otimes X - \sigma_1 \otimes Y, \quad M^T = M, \qquad (C.2)$$

where σ_1, σ_3 are the Pauli matrices (A.4). It is known that any symmetric real matrix is diagonalized by using a real orthogonal matrix O:

$$O^T M O = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}), \quad O^T O = I_{2n}, \qquad (C.3)$$

where I_{2n} denotes the $2n$-dimensional unit matrix. Obviously, all diagonal elements λ_k are real numbers. Taking into account $(O^T)^{-1} = O$, relations (C.3) are written in the components as

$$M_{ik} O_{k\ell} = O_{i\ell} \lambda_\ell, \quad O_{ki} O_{k\ell} = \delta_{i\ell}. \qquad (C.4)$$

Now we introduce the set of real $2n$-dimensional vectors $v^{(\ell)}$ with the coordinates $v_i^{(\ell)} = O_{i\ell}$ (i.e., the vectors $v^{(\ell)}$ are the columns of the matrix O). From the relations (C.4) it becomes clear that $v^{(\ell)}$ are eigenvectors of the matrix M with eigenvalues λ_ℓ:

$$M v^{(\ell)} = (\sigma_3 \otimes X - \sigma_1 \otimes Y) v^{(\ell)} = \lambda_\ell \&^{(\ell)}, \qquad (C.5)$$

where the set of $2n$ vectors $v^{(\ell)}$ forms an orthonormal system: $(v^{(\ell)}, v^{(i)}) = \delta_{\ell i}$, and therefore defines a basis in $\mathbb{R}^{2n}$.

Now we note that if λ_ℓ is an eigenvalue of M, then $-\lambda_\ell$ is also an eigenvalue of M. Indeed, let us multiply both sides of relation (C.5) from the left by the non-singular matrix

$$B := i(\sigma_2 \otimes I_n) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and use the obvious relation $BM = -MB$. As a result we obtain $M(Bv^{(\ell)}) = -\lambda_\ell (Bv^{(\ell)})$, i.e. $Bv^{(\ell)}$ is an eigenvector of the matrix M with an eigenvalue $-\lambda_\ell$. Thus, all $2n$ eigenvalues of the matrix M are divided into pairs $(\lambda_\ell, -\lambda_\ell)$, and each pair has one obviously non-negative eigenvalue. We choose all such non-negative eigenvalues[3], of which there are n, and denote these eigenvalues as $m_1, m_2, \ldots m_n$, while the corresponding $2n$-dimensional real eigenvectors are denoted as $V^{(1)}, \ldots, V^{(n)}$:

$$M V^{(k)} = m_k V^{(k)} \Rightarrow \begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix} \begin{pmatrix} u^{(k)} \\ w^{(k)} \end{pmatrix} = m_k \begin{pmatrix} u^{(k)} \\ w^{(k)} \end{pmatrix}, \qquad (C.6)$$

where $k = 1, \ldots, n$, and we composed $2n$-dimensional vectors $V^{(k)}$ of two ndimensional vectors $u^{(k)}$ and $w^{(k)}$. Note that the orthonormality property for any selection of vectors $V^{(k)}$ is preserved, and we have

$$\delta_{k\ell} = (V^{(k)}, V^{(\ell)}) = (u^{(k)}, u^{(\ell)}) + (w^{(k)}, w^{(\ell)}). \qquad (C.7)$$

Let us introduce n-dimensional complex vectors $z^{(k)} = u^{(k)} + i w^{(k)}$, $k = 1, \ldots, n$. Then the second equality in (C.6) and the orthonormality condition (C.7) can be written as

$$M_{\mathbb{C}} z^{(k)} = m_k z^{(k)*}, \quad (z^{(k)*}, z^{(\ell)}) = \delta_{k\ell}, \qquad (C.8)$$

where $z^{(k)*} = u^{(k)} - i w^{(k)}$ and $m_k \in \mathbb{R}_{\geq 0}$. Now we define the complex $n \times n$-matrix U, the columns of which are the vectors $z^{(k)}$, i.e. $U_{ik} = z_i^{(k)}$. Then the relations (C.8) are presented in the form

$$M_{\mathbb{C}} U = U^* \,\mathrm{diag}(m_1, \ldots, m_n), \quad U^\dagger U = I_n.$$

Thus, we have constructed a unitary matrix $U \in U(n)$, which diagonizes, according to (C.1), the complex symmetric $n \times n$-matrix $M_{\mathbb{C}}$, and, in addition, the diagonal elements m_k are non-negative real numbers, as required.

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[3] If both eigenvalues in a pair are equal to zero, then either of them is chosen.

dimensional space with n additional spatial dimensions compactified on the length scale L, which corresponds to the energy scale $\Lambda_L = 1/L$. Integration over additional dimensions leads to the Lagrangian of the low-energy effective field theory, which after spontaneous symmetry breaking gives the Majorana mass term for active neutrinos with small masses of the order $m_\nu \sim m_D^2/\Lambda_L \sim 10^{-3}$ eV, where the scale is taken to be $\Lambda_L \sim 100$ TeV.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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