ISSN 0040-5795, Theoretical Foundations of Chemical Engineering, 2019, Vol. 53, No. 6, pp. 967–971.

Semiempirical Method for Determining the Rate of Slow Hindered Motion

A. M. Trushin[a], L. V. Ravichev[a], V. E. Yashin[a], M. A. Nosyrev[a, *], and R. B. Komlyashev[a]

[a] *Mendeleev University of Chemical Technology of Russia, Moscow, 125047 Russia*

**e-mail: vnissok@list.ru*

Received May 20, 2019; revised May 29, 2019; accepted June 3, 2019

Abstract—Semiempirical equations have been derived for determining the rate of slow hindered motion of spherical particles in a liquid, which agree with empirical data with a high degree of accuracy. It has been shown that the variational principle of the minimum intensity of energy dissipation can be used in determining the rate of hindered motion.

Keywords: rate of slow hindered motion, spherical particles, variational principle of the minimum intensity of energy dissipation

DOI: 10.1134/S0040579519060125

INTRODUCTION

Determining the rate of hindered motion of dispersed particles, as well as solving the similar problem of finding the velocity of a fluidizing agent in a fluidized bed, is performed by analytical, semiempirical, and empirical methods.

A purely analytical approach was used in cell models [1]. Methods based on the theory of effective viscosity and methods based on the determination of resistance coefficients are classified among semiempirical methods. In a number of studies [2–4], cell models and models based on resistance coefficients were developed using the variational principle of the minimum intensity of energy dissipation.

It should be noted that analytical and semiempirical computational methods yield large errors in the region of the high concentrations of the dispersed phase. The exception is the Happel cell model, the error of calculation by which reaches 100%, but at very high concentrations of the solid phase that verge on a fixed bed, and the results of calculation approximate to experimental data. This can be explained by the fact that the imperfection of the model itself is compensated for by the error of calculation at high concentrations of dispersed particles.

Highly concentrated disperse systems occur in fluidized beds, in sedimentation processes, and in the rise of an ensemble of bubbles [5].

The objective of this study is to determine the rate of slow hindered settling of spherical particles and prove that the rate of hindered motion is the consequence of the self-organization of a system of dispersed particles and can be determined from the variational principle of the minimum intensity of energy dissipation, as was assumed in [2–4].

HINDERED MOTION OF PARTICLES AT HIGH CONCENTRATIONS OF THE DISPERSED PHASE

We consider the ideal case of the motion of a liquid in the channels of a uniform fluidized bed that are assumed to be straight with the constant equivalent radius R_e. We also assume that the velocity of the liquid near the surface of spherical monodisperse solid particles is zero. The pressure gradient in the fluidized bed with the fraction of the dispersed phase φ is equal to the gravity of the unit volume of the fluidized bed with the sign reversed:

$$\nabla p = -[\rho_s\varphi + \rho(1-\varphi)]g. \quad (1)$$

With this taken into account, the balance of external and internal forces per unit volume of the liquid flowing in the channels of the fluidized bed is expressed by the following equation in cylindrical coordinates:

$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_{id}}{dr}\right) = -(\rho_s - \rho)\varphi g. \quad (2)$$

Equation (2) serves as a basis for determining the dependence of the rate of hindered motion of particles on the fraction of the dispersed phase.

It is known that the Hagen–Poiseuille equation can be derived using the variational method.

Since one of the objectives of this study is to prove that the rate of slow hindered motion can be determined by the variational method and in view of the fact that the motion of a liquid through the channels of a fluidized bed has specific features, it is necessary to show that the principle of the minimum intensity of energy dissipation can be applied in this case. To accomplish this, it is sufficient to show that Eq. (2) corresponds to the functional that gives the minimum intensity of dissipation.

The functional in the form of the intensity of dissipation under the conditions of laminar motion in the channels of a fluidized bed with the length L can be written as follows:

$$I(V_{\mathrm{id}}) = 2\pi\int_0^{R_\mathrm{e}} \mu L V_{\mathrm{id}}\left(\frac{\mathrm{d}V_{\mathrm{id}}}{\mathrm{d}r} + r\frac{\mathrm{d}^2V_{\mathrm{id}}}{\mathrm{d}r^2}\right)\mathrm{d}r = \min. \quad (3)$$

The function $V_{\mathrm{id}}(r)$ that gives the extremum of the functional $I(V_{\mathrm{id}})$ is derived from the Euler–Lagrange equation:

$$\frac{\partial F}{\partial V_{\mathrm{id}}} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial F}{\partial V'_{\mathrm{id}}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{\partial F}{\partial V''_{\mathrm{id}}}\right) = 0, \quad (4)$$

where F is the integrand of the functional.

At a constant flow rate of the liquid, there is the following additional condition for variational problem (3):

$$2\pi\int_0^{R_\mathrm{e}} V_{\mathrm{id}}(\rho_\mathrm{s} - \rho)\varphi g L r\,\mathrm{d}r = \mathrm{const}. \quad (5)$$

With allowance for additional condition (5), the function F in Eq. (4) is replaced with the function Φ:

$$\Phi = \mu L V_{\mathrm{id}}\left(\frac{\mathrm{d}V_{\mathrm{id}}}{\mathrm{d}r} + r\frac{\mathrm{d}^2V_{\mathrm{id}}}{\mathrm{d}r^2}\right) + \lambda V_{\mathrm{id}}(\rho_\mathrm{s} - \rho)g\varphi L r, \quad (6)$$

where λ is the Lagrangian multiplier.

In this case, from the Euler–Lagrange equation, we derive the following second-order differential equation:

$$\mu\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}V_{\mathrm{id}}}{\mathrm{d}r}\right) = -\frac{\lambda}{2}(\rho_\mathrm{s} - \rho)g\varphi. \quad (7)$$

To observe the balance of external and internal forces per unit volume of the liquid, λ in Eq. (7) must have a value of 2, which corresponds to the complete identity of Eqs. (2) and (7). Thus, it can be considered that the problem of finding the superficial velocity of a liquid in a uniform fluidized bed or the equivalent problem of finding the rate of hindered settling, the solving of which is based on Eq. (2), can be solved by the variational method.

Solving Eq. (2) with the subsequent determination of the volumetric flow rate of the liquid through the cross section of the channel of the fluidized bed leads to the following relationship for the average velocity of the liquid passing through the bed in the ideal case:

$$V_{\mathrm{id,avg}} = R_\mathrm{e}^2\frac{(\rho_\mathrm{s} - \rho)g\varphi}{8\mu} = d_\mathrm{e}^2\frac{(\rho_\mathrm{s} - \rho)g\varphi}{32\mu}. \quad (8)$$

In the case of a granular bed of spherical particles with diameter d, the equivalent diameter of virtual cylindrical channels is calculated using the equation

$$d_\mathrm{e} = \frac{2}{3}\frac{(1-\varphi)}{\varphi}d. \quad (9)$$

From Eqs. (8) and (9), we derive the following relationship:

$$V_{\mathrm{id,avg}} = \frac{d^2(\rho_\mathrm{s} - \rho)g(1-\varphi)^2}{72\mu\varphi}. \quad (10)$$

The average velocity of the liquid is related to the superficial velocity $V_{\mathrm{id,sup}}$ by the equation

$$V_{\mathrm{id,avg}} = \frac{V_{\mathrm{id,sup}}}{1-\varphi}. \quad (11)$$

Taking into account (11), we derive the following relationship:

$$V_{\mathrm{id,sup}} = \frac{d^2(\rho_\mathrm{s} - \rho)g(1-\varphi)^3}{72\mu\varphi}. \quad (12)$$

It should be noted that a similar approach was used by Kozeny, Carman, and Blake [6] for deriving the pressure gradient in a fixed granular bed based on the Hagen–Poiseuille equation.

We now write Eq. (12) in the dimensionless form

$$\mathrm{Re}_{\mathrm{id}} = \frac{\mathrm{Ar}}{72}\frac{(1-\varphi)^3}{\varphi}. \quad (13)$$

In view of the fact that the channels of a fluidized bed are tortuous and have a nonuniform cross section and that particles in the fluidized bed can rotate, the actual velocity V_{sup} differs from the ideal velocity $V_{\mathrm{id,sup}}$:

$$V_{\mathrm{sup}} = \frac{V_{\mathrm{id,sup}}}{K}; \quad K > 1. \quad (14)$$

Taking into account (14), we derive the following equation for the actual superficial velocity:

$$\mathrm{Re} = \frac{\mathrm{Ar}}{72K}\frac{(1-\varphi)^3}{\varphi}. \quad (15)$$

From analogy between fluidization and settling, it is known that the actual rate of hindered settling of particles is equal to the superficial velocity of a liquid in a fluidized bed V_{sup}. Therefore, Eq. (15) can be used to determine the rate of hindered settling.

Since Stokes' law is valid at small values of Re (slow flow), Eq. (15) can be written for this case as follows:

$$\mathrm{Re} = \frac{\mathrm{Ar}}{18}f(\varphi). \quad (16)$$

The function $f(\varphi) = (1-\varphi)^3/(4K\varphi)$ is the ratio of hindered settling velocity to free settling velocity.

We write $f(\varphi)$ in the following simplified form:

$$f(\varphi) = A\frac{(1-\varphi)^3}{\varphi}, \tag{17}$$

where $A = 1/(4K)$.

The complexity of the problem of the motion of a liquid through tortuous channels with a variable cross-sectional area necessitates using experimental data for finding the value of A.

The ratio of the hindered settling velocity to the free settling velocity of fine-dispersed spherical particles in the laminar region is determined with a sufficient degree of accuracy using power-law functions [7]:

$$f(\varphi) = (1-\varphi)^n, \tag{18}$$

$$n = 4.65 + 19.5\frac{d}{d_{\mathrm{col}}} \quad \text{at} \quad \mathrm{Re}_0 < 0.2, \tag{19}$$

$$n = \left(4.35 + 17.5\frac{d}{d_{\mathrm{col}}}\right)\mathrm{Re}_0^{-0.03} \quad \text{at} \quad 0.2 < \mathrm{Re}_0 < 2. \tag{20}$$

The Reynolds number Re_0 is determined from free settling velocity.

Since the values of $\mathrm{Re} < 2$ correspond to the diameters of particles that differ from the diameter of columns d_{col} by several orders of magnitude, the quantities $19.5d/d_{\mathrm{col}}$ and $17.5d/d_{\mathrm{col}}$ were not taken into account in calculating the value of n.

The values of coefficients A that give the closest agreement between Eqs. (17) and (18) at $\mathrm{Re}_0 < 0.2$ were calculated using the following system of equations:

$$J = \int_{1-\varphi_1}^{1-\varphi_2}\left(\varepsilon^n - A\frac{\varepsilon^3}{1-\varepsilon}\right)^2 \mathrm{d}\varepsilon = \min, \tag{21}$$

$$\frac{\mathrm{d}J}{\mathrm{d}A} = -2\int_{1-\varphi_1}^{1-\varphi_2}\left[\left(\varepsilon^n - A\frac{\varepsilon^3}{1-\varepsilon}\right)\left(\frac{\varepsilon^3}{1-\varepsilon}\right)\right]\mathrm{d}\varepsilon = 0. \tag{22}$$

For convenience of calculation, the fraction of the liquid phase $\varepsilon = 1 - \varphi$ was used in Eqs. (21) and (22).

Integral (17) was calculated between the limits $\varphi_1 = 0.25$ and $\varphi_2 = 0.55$. The upper limit $\varphi_2 = 0.55$ is close to a fixed bed. As calculations have shown, at particle concentrations lower than $\varphi_1 = 0.25$, discrepancies between Eqs. (17) and (18) become considerable. This is probably associated with the fact that, at low concentrations of the solid phase, channels through which a liquid flows break down and, accordingly, the proposed model of the motion of dispersed particles is bounded by the range of the fraction of the dispersed phase from 0.25 to 0.55. Equations (16) and (17) are applicable for the values of $\mathrm{Re}_0 < 0.2$, when Stokes' law gives fairly small disagreement with experimental data. In the case of slow motion ($0 < \mathrm{Re}_0 < 0.2$), the value of coefficient A calculated by Eq. (22) is 0.16.

Thus, for the slow motion of spherical particles, the ratio between the rates of hindered and free motion in the range of the fractions of the dispersed phase from 0.25 to 0.55 can be determined as follows:

$$f(\varphi) = 0.16\frac{(1-\varphi)^3}{\varphi}. \tag{23}$$

The mean difference between the values calculated by Eqs. (18) and (23) is 4.7%.

HINDERED MOTION AT LOW VALUES OF THE FRACTION OF THE DISPERSED PHASE

The hindered settling velocity of spherical particles at low concentrations of the dispersed phase in the interval of $0 < \varphi < 0.25$ at $\mathrm{Re}_0 < 0.2$ can be determined based on the representation of a suspension as a Newtonian fluid, which has a higher (effective) viscosity μ_{eff}, which is the function of the fraction of the dispersed phase.

The dimensionless effective viscosity $\bar{\mu}$, which is the ratio between the viscosity of a suspension and the viscosity of a dispersion medium, can be determined over a fairly wide range of the fractions of the dispersed phase ($0 < \varphi < 0.4$) using the following experimentally found formula [8]:

$$\bar{\mu} = 1 + 2.5\varphi + 12.5\varphi^2. \tag{24}$$

At low values of φ, formula (24) transforms into the Einstein equation.

Strongly approximate consideration of the settling of a foreign spherical particle in a medium with the effective viscosity and density of a suspension in calculating the buoyancy force leads to the following ratio between the rates of hindered and free settling of spherical particles in the range of $\mathrm{Re}_0 < 0.2$:

$$f(\varphi) = \frac{(1-\varphi)^2}{1 + 2.5\varphi + 12.5\varphi^2}. \tag{25}$$

In deriving Eq. (25), it was assumed that the resistance coefficient in hindered settling in a medium with effective viscosity and in a dispersion medium complies with Stokes' law.

It follows from [3] that this equation yields underestimated results, which is associated with the overestimated value of the buoyant force. It should be noted that this expression of the buoyant force contradicts the assumption of the uniformity of the motion of particles in settling. The elimination of this contradiction leads to the weaker dependence of $f(\varphi)$ on the fraction of a continuous phase:

$$f(\varphi) = \frac{1-\varphi}{1 + 2.5\varphi + 12.5\varphi^2}. \tag{26}$$

In turn, Eq. (26) corresponds to the underestimated values of $f(\varphi)$ when compared to empirical data.

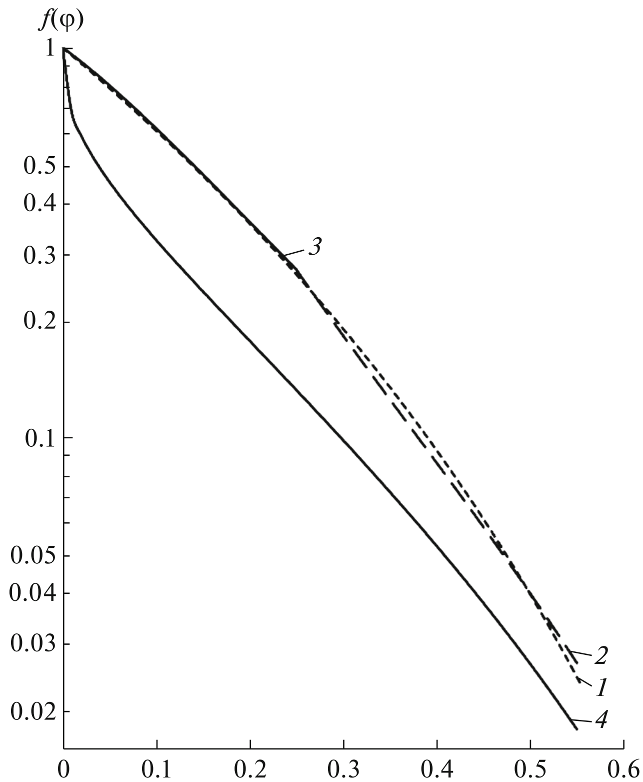

Fig. 1. Dependence of the ratio between the hindered and free settling velocities of particles on the fraction of the dispersed phase: (*1*) formula (18), (*2*) formula (23), (*3*) formulas (27) and (29), and (*4*) Happel equation.

Calculations have shown that the geometric average from Eqs. (25) and (26) gives the smallest deviation from empirical data:

$$f(\varphi) = \frac{(1-\varphi)^{1.5}}{1 + 2.5\varphi + 12.5\varphi^2}. \quad (27)$$

The mean difference between the values of $f(\varphi)$ calculated using Eqs. (27) and (18) is 1.2%.

The same result can be derived from the interrelationship between $f(\varphi)$ and dimensionless effective viscosity [8]:

$$\bar{\mu} = \frac{(1-\varphi)^m}{f(\varphi)}. \quad (28)$$

The two values of the power m are used in this formula: $m = 1$ (the Kynch relationship) and $m = 2$ (the Hawksley relationship). Here, the function $f(\varphi)$ is calculated using the cell model for the purpose of finding effective viscosity. If the geometric average for the function $(1-\varphi)^m$ at values of $m = 1$ and $m = 2$ is taken, we have formula (27) as a result, with formula (24) being used to calculate effective viscosity.

Equations (25)–(28) were derived based on the uniform distribution of particles throughout the bed, whereas, in settling, the fraction of the dispersed phase is nonuniformly distributed due to the integration of particles into groups [2–4]. In this case, the true fraction of the dispersed phase is greater than the average fraction. If we assume that the true value φ_{true} exceeds the average value φ by 12.5%, formula (26) transforms into the following equation:

$$f(\varphi) = \frac{1 - 1.125\varphi}{1 + 2.813\varphi + 15.82\varphi^2}. \quad (29)$$

The mean difference between the values of $f(\varphi)$ calculated using Eqs. (27) and (29) is less than 1%. Therefore, it can be assumed that the cause of the overestimated values of $f(\varphi)$ calculated using Eq. (26) is the integration of particles into groups in settling.

As can be seen from Fig. 1, where empirical equation (18) is compared with semiempirical dependences (23), (27), and (29) and the Happel theoretical equation [6], the derived semiempirical dependences are very close to the empirical equation, whereas the Happel theoretical equation deviates to a considerable extent from both empirical and semiempirical dependences. Taking into account the higher accuracy of empirical equation (18) [9], we can infer that theoretical approaches are sufficiently valid in the derivation of semiempirical relationships for the ratio between the rates of hindered and free settling.

CONCLUSIONS

Summing up, we can infer that the problem of determining the rate of slow hindered settling is solved using Eq. (23) for high values of the fraction of the dispersed phase ($0.25 < \varphi < 0.55$) and Eq. (27) or (29) in the region of low concentrations of dispersed particles ($0 < \varphi < 0.25$). It can also be inferred that slow hindered motion obeys the variational principle of the minimum intensity of energy dissipation.

NOTATION

d	particle diameter
d_e	equivalent diameter of channels
g	acceleration due to gravity
K	correction factor
R_e	equivalent radius of channels
V_{id}	velocity of the liquid in channels in the ideal case
$V_{id,sup}$	superficial velocity of the liquid in the ideal case
V_{sup}	superficial velocity of the liquid
ε	fraction of the liquid phase
λ	Lagrangian multiplier
μ	dynamic viscosity
$\bar{\mu}$	reduced effective viscosity
ρ	liquid density

ρ_s	particle density
φ	fraction of the dispersed phase
Re	Reynolds number for the hindered motion of particles
Re_0	Reynolds number for the free motion of particles
Re_{id}	Reynolds number in the ideal case of motion in channels

REFERENCES

1. Slobodov, E.B. and Chepura, I.V., On the cell model of two-phase media, *Teor. Osn. Khim. Tekhnol.*, 1982, vol. 16, no. 3, p. 331.
2. Trushin, A.M., Dmitriev, E.A., Nosyrev, M.A., Tarasova, T.A., and Kuznetsova, I.K., Determining the velocity of the hindered motion of spherical gas particles through liquid in a gravity field, *Theor. Found. Chem. Eng.*, 2013, vol. 47, no. 4, pp. 368–374. https://doi.org/10.1134/S0040579513040337
3. Trushin, A.M., Dmitriev, E.A., Nosyrev, M.A., Khusanov, A.E., and Kaldybaeva, B.M., General method of measurement of velocity of laminar constrained motion of spherical solid and gas particles in liquids, *Theor. Found. Chem. Eng.*, 2013, vol. 47, no. 6, pp. 730–733. https://doi.org/10.1134/S0040579513060110
4. Trushin, A.M., Dmitriev, E.A., and Nosyrev, M.A., The use of the variation method to define the velocity of constrained turbulent motion of disperse particles in liquid, *Theor. Found. Chem. Eng.*, 2017, vol. 51, no. 2, pp. 155–158. https://doi.org/10.1134/S0040579517020117
5. Nosyrev, M.A., Trushin, A.M., Komlyashev, R.B., Kabanov, O.V., and Kuznetsova, I.K., Biogas purification in a microbubble column, *Khim. Prom-st. Segodnya,* 2018, no. 3, p. 32.
6. Protod'yakonov, I.O. and Chesnokov, Yu.G., *Gidromekhanicheskie osnovy protsessov khimicheskoi tekhnologii* (Hydromechanical Fundamentals of Processes in Chemical Technology), Leningrad: Khimiya, 1987.
7. Shvydkii, V.S., *Mekhanika zhidkosti i gaza* (Fluid Mechanics), Moscow: Akademkniga, 2003.
8. Kutepov, A.M., Polyanin, A.D., Zapryanov, Z.D., Vyaz'min, A.V., and Kazenin, D.A., *Khimicheskaya gidrodinamika* (Chemical Fluid Dynamics), Moscow: Byuro Kvantum, 1996.
9. *Fluidization,* Davidson, J.F. and Harrison, D., Eds., London: Academic, 1971.

Translated by A. Uteshinsky