Polyhedral methodology for conflict control of competing objects in pursuit conditions

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***Abstract*—A polyhedral methodology for optimization of discrete time control processes is exploited. A problem of polyhedral discrete time dynamic game of pursuit is defined. A polyhedral strategy of pursuit based on the concept of a guaranteed blunder is proposed.**

I. Elements of the polyhedral optimization theory

In the last years development of methods for analysis and synthesis of control systems based on nonsmooth analysis [1], [2] gains popularity in the theory and application of automatic control. Thus, in the context of nondifferentiable optimization of control processes the polyhedral optimization methodology [3], [4] appears quite promising. It targets two cardinal problems of modern automation: account of direct measures of control process quality and account of direct resource and phase constraints.

Consider a class of dynamic controllable objects described by a linear difference equation of state vectors in the form of

$$x(t+1) = A(t)x(t) + B(t)u(t), \qquad (1)$$

where $t \in \mathcal{T}$ is discrete time, $\mathcal{T} = [0 : T-1] \subset \mathbb{Z}_+$ is the interval of control, $T \geq 1$ is the final (terminal) time instant, $x = col(x_1, x_2, \ldots, x_n) \in \mathbb{X} \subset \mathbb{R}^n$ is a column vector of state variables, $u = col(u_1, u_2, \ldots, u_r) \in \mathbb{U} \subset \mathbb{R}^r$ is a column vector of controlling variables, $A : \mathcal{T} \to \mathbb{R}^{n\times n}$ and $B : \mathcal{T} \to \mathbb{R}^{n\times r}$ are functional matrices, $\mathbb{X}$ is state space, $\mathbb{Z}_+$ is the nonnegative integer set, $\mathbb{R}^i$ is i-dimensional real linear space.

A. Main concepts of polyhedral formalism

First of all we consider two designs of convex analysis [3]: polyhedral functions and polyhedral norms.

A *polyhedral function* $f(x) : \mathbb{X} = \mathbb{R}^n \to \mathbb{R}$ is a function, the epigraph of which is a convex polyhedron. The most important constructive property of any polyhedral function $f(x)$ is the possibility of its disjunctive expansion. That is its representation in the form of a function of discrete (pointwise) maximum:

$$f(x) = \bigvee_{i=1}^{N} \varphi_i(x) = \max\{\varphi_1(x), \varphi_2(x), \ldots, \varphi_N(x)\},$$

where all $\varphi_i(x)$ are affine functions

$$\varphi_i(x) = a_{i0} + \langle a_i, x\rangle, \quad a_{i0} \in \mathbb{R},\ a_i \in \mathbb{X}.$$

A *polyhedral norm* of a vector is a polyhedral function of its components. For example, *cubic* (*Chebyshev*) norm:

$$||x||_\infty = \max_{1\leq i\leq n} |x_i|.$$

B. Formalization of polyhedral quality criteria for control processes

It is clear that the requirements for any dynamic structure of trajectories of controllable object movement and also for resources (cost) of controlling actions, necessary for the realization of this movement, must be represented by the structure of the quality criterion.

Let x^* denotes the goal state of the controlled object. We'll need the following values:

$$\varepsilon(t) = x(t) - x^*, \quad \Delta x(t) = x(t+1) - x(t),$$
$$\Delta u(t) = u(t+1) - u(t).$$

They characterize respectively: distance of the object's state to the goal state, phase speed of the object and intensity of controlling action at current time instant. Also, choose polyhedral norms:

$$\mathcal{H}_\varepsilon : \mathbb{X} \to \mathbb{R},\ \mathcal{H}_{\Delta x} : \mathbb{X} \to \mathbb{R},\ \mathcal{H}_u : \mathbb{U} \to \mathbb{R},\ \mathcal{H}_{\Delta u} : \mathbb{U} \to \mathbb{R}.$$

The quality of a control process at the current time instant we define by measures of control exactness and inputs of control. They have a polyhedral structure and are formulated by a combination of the values $\mathcal{H}_\varepsilon(\varepsilon(t))$, $\mathcal{H}_{\Delta x}(\Delta x(t))$, $\mathcal{H}_u(u(t))$, $\mathcal{H}_{\Delta x}(\Delta x(t))$. We shall exemplify these parameters of quality with the following notations:

- polyhedral measures of control accuracy

$$\mathcal{P}(t) = \lambda_\varepsilon(t)\mathcal{H}_\varepsilon(\varepsilon(t)) + \lambda_{\Delta x}(t)\mathcal{H}_{\Delta x}(\Delta x(t));$$
$$\mathcal{P}(t) = \max\left\{\lambda_\varepsilon(t)\mathcal{H}_\varepsilon(\varepsilon(t)); \lambda_{\Delta x}(t)\mathcal{H}_{\Delta x}(\Delta x(t))\right\}.$$

- polyhedral measures of control inputs

$$\mathcal{Q}(t) = \lambda_u(t)\mathcal{H}_u(u(t)) + \lambda_{\Delta u}(t)\mathcal{H}_{\Delta u}(\Delta u(t));$$
$$\mathcal{Q}(t) = \max\left\{\lambda_u(t)\mathcal{H}_u(u(t)), \lambda_{\Delta u}(t)\mathcal{H}_{\Delta u}(\Delta u(t))\right\}.$$

Here $\lambda_\varepsilon(t)$, $\lambda_{\Delta x}(t)$, $\lambda_u(t)$, $\lambda_{\Delta u}(t)$ are nonnegative weight coefficients. They in particular may have the form of exponential functions: $c_v t^v$, $v \in \mathbb{Z}_+$, $c_v \in \mathbb{R}$.

Notice that the polyhedral measures $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are related to discrete time t, therefore they are discrete themselves.

From the introduced accuracy and resource measures we may form different polyhedral criteria of quality of control processes. Let $\mathcal{T}^+ = [1:T]$. Consider the following examples:

- polyhedral terminal criterion (Mayer type)

$$\mathcal{F}_M = \mathcal{P}(T);$$

- polyhedral integral criteria (Lagrangian type)

$$\mathcal{F}_L = \sum_{t=1}^{T} \mathcal{P}(t) + \sum_{t=0}^{T-1} \mathcal{Q}(t);$$

$$\mathcal{F}_L = \max_t \left\{ \{\mathcal{P}(t), t \in \mathcal{T}^+\} \right\} + \max_t \left\{ \{\mathcal{Q}(t), t \in \mathcal{T}\} \right\};$$

$$\mathcal{F}_L = \max_t \left\{ \{\mathcal{P}(t), t \in \mathcal{T}^+\} \cup \{\mathcal{Q}(t), t \in \mathcal{T}\} \right\};$$

- polyhedral mixed criterion (Bolza type)

$$\mathcal{F}_B = \mathcal{F}_M + \mathcal{F}_L.$$

The appearing values $x(T+1)$ and $u(t)$ in the above criteria are not a part of the mathematical model of controllable object (1). Formally we may define them as:

$$x(T+1) = x(T), \quad u(T) = u(T-1).$$

Let us give an example of the mentioned criteria which is very promising for optimal stabilization problems. Assume that the purpose of control is stabilization of an object into its equilibrium state $x^* = \mathbf{0}$. Also it is necessary for any (deviated) starting state $x(0) \equiv x_0 \neq \mathbf{0}$ the object to be quited down in a final (terminal) time instant $t = T : x(T) = \mathbf{0}$.

Let $u(t) = u(t, x_0)$ and $x(t) = x(t, x_0)$ be the current values of control and state of the controlled object in its motion from the starting state x_0, and $\Delta u(t)$ and $\Delta x(t)$ be the speeds of their change respectively.

In order to estimate the quality of a stabilization process we offer to use the following (introduced in in the works [3], [4]) *polyhedral loss function*, taking into account the dynamic structure of phase trajectories and controlling actions:

$$\mathcal{F} = \max_{0 \le t \le T-1} \mathcal{Q}(x(t), \Delta x(t), u(t), \Delta u(t)). \tag{2}$$

Here $\mathcal{Q}(x, \Delta x, u, \Delta u)$ is a polyhedral measure of control cost

$$\begin{aligned}\mathcal{Q}(x, \Delta x, u, \Delta u) = \quad & \lambda_1(t) q_1(x(t)) + \lambda_2(t) q_2(\Delta x(t)) + \\ & \lambda_3(t) q_3(u(t)) + \lambda_4(t) q_4(\Delta u(t)),\end{aligned}$$

where $q_1 : \mathbb{R}^n \to \mathbb{R}$, $q_2 : \mathbb{R}^n \to \mathbb{R}$, $q_3 : \mathbb{R}^r \to \mathbb{R}$, $q_4 : \mathbb{R}^r \to \mathbb{R}$ are some positive homogeneous polyhedral functions, and all $\lambda_i(t)$ are nonnegative weight coefficients that yield $\sum_{i=1:4} \lambda_i(t) > 0$ for all $t = 0 : T-1$.

C. Polyhedral Chebyshev criterion

Consider another example of a polyhedral criterion for quality of stabilization processes. Here we have the same goal for the control process — stabilization of an object into its equilibrium point $x* = \mathbf{0}$. As a degree of deviation from the equilibrium state of an object we take the polyhedral measure

$$\mathcal{P}(t) = \mathcal{H}_x(x(t)) = ||x(t)||_\infty = \max_{i \in 1:n} |x_i(t)|.$$

Then the criterion for quality of a stabilization process is the maximum value of $\mathcal{P}(t)$ on the interval, in which the system functions:

$$\mathcal{F} = \max_t \{\mathcal{P}(t)\} = \max_{0 \le t \le T-1} ||x(t)||_\infty. \tag{3}$$

Criterion (3) finds the *maximum dynamic mistake* (that is the maximum amplitude of all state variables) in a stabilization system and is known as *criterion of uniform approach*, *maximum deviation* and *Chebyshev's criterion*.

The most successful attempts in applying the polyhedral criterion for quality (3) were executed in the 1060s by A. A. Pervozvansky. He worked on uniform optimization of control processes (see [6]). Later many honored scientists repeatedly confirmed the special importance of Chebyshev's criterion regarding applied control problems. We ought to outline some of the russian speaking authors: E. A. Barbashin, N. N. Krasovskii, A. B. Kurzhanskii, Ju. S. Osipov, Ja. Z. Cypkin, K. A. Lurie, V. A. Jakubovich, V. F. Demyanov, V. M. Kein, B. T. Polyak, R. Gabasov, F. M. Kirillova, V. A. Troickii, A. G. Chencov, A. E. Barabanov, O. N. Granichin, S. F. Sokolov and others.

D. Formalization of the polyhedral optimization problem

The process of controlling an object of type (1) is a union of control influence-program application $U = \{u(t), t \in \mathcal{T}\}$ and its generated phase motion trajectory $X = \{x(t), t \in \mathcal{T}^+\}$.

Suppose that the control influence implements a *program-position control strategy* as flexible, cyclically updated programs.

The general problem of polyhedral optimization of discrete control processes consists in finding a control influence $u(t)$ for object (1) such that the terminal goal of control is achieved

$$x(T) \in \mathbb{X}^* = \{x \in \mathbb{X} : H(x) \le p^*\}, \quad H : \mathbb{X} \to \mathbb{R}, \ p^* \in \mathbb{R},$$

and u(t) is optimal in the terms of criterion

$$F(X, U) \to extr, \quad F : \mathbb{X} \times \mathbb{U} \to \mathbb{R},$$

with respect to phase and resource constraints

$$\begin{gathered} P(x(t)) \le p(t), \quad E(u(t)) \le q(t), \\ t \in \mathcal{T}, \ p(t) > 0, \ q(t) > 0, \ P : \mathbb{X} \to \mathbb{R}, \ E : \mathbb{U} \to \mathbb{R}. \end{gathered}$$

The specifics of this problem are such that all of its components (terminal goal of control, optimality criterion, phase and resource constraints) are of polyhedral structure — i.e. the functions $H(x)$, $P(x)$, $E(u)$, $F(X, U)$ are polyhedral. Therefore, we face the class of *polyhedral programming problems* [4]. The algorithms that cope with the latter are based on linear programming computational methods.

The polyhedral methodology is a base that allows successful solving of many key control processes optimization problems, including optimization problems of observation and control processes under conditions of uncertainty about regular, conflict and critical situations [4].

As an example we will address a problem of polyhedral conflict control of competing objects in pursuit conditions.

II. Polyhedral methodology for game control optimization in pursuit conditions

Many applied control problems source from ones that are objects of study in the pursuit game theory. One of the first to address and thoroughly study the pursuit problem was D. L. Kelendzheridze in 1961. Fundamental results in the field of dynamic pursuit games have been obtained in the works of R. Isaacs, A. E. Bryson, W. H. Fleming, A. Friedman, N. N. Krasovskii, A. I. Subbotin, A. G. Chencov, V. E. Tretiakov, V. M. Kein, L. S. Pontriagin, E. F. Mishenko, M. S. Nikolskii, B. N. Pshenichny, F. L. Chernousko, L. A. Petrosjan, O. A. Malafeev, G. V. Tomskii, V. I. Zubov, M. I. Zelikin, E. N. Simakova, E. M. Vaisbord, V. I. Zhukovskii, N. L. Grigorenko and others [7], [8], [9], [10], [11].

A. Discrete dynamic pursuit games

In this section we will address one class of discrete dynamic pursuit games [12], [13]. Suppose that two competing players are described as moving objects. Consider a discrete *interception game* in which the first player P (Pursuer) tries to intercept the second player E (Evader). The process of this game is described with a linear difference equation

$$x(t+1) = Ax(t) + u(t) + v(t), \tag{4}$$

where $t \in \mathcal{T}$ is discrete time, $\mathcal{T} \subset \mathbb{Z}_+$ is the time interval of the game process, $x \in \mathbb{X}$ is a *game position* — a n-dimensional vector of player P's relative coordinates in a system connected to player E, $\mathbb{X} = \mathbb{R}^n$ is game space, $u \in \mathbb{U}$ and $v \in \mathbb{V}$ are n-dimensional control vectors of players P and E respectively, $\mathbb{U} \subset \mathbb{R}^n$ and $\mathbb{V} \subset \mathbb{R}^n$ are sets of control, $A \in \mathbb{R}^{n \times n}$.

Suppose the sets of control $\mathbb{U}$ and $\mathbb{V}$ are polyhedral and both players possess full information about the current game position — i.e. vector x.

The strategies of players P and E we denote by ξ and η respectively:

$$\xi = u(\cdot) \in \mathcal{U}, \quad \eta = v(\cdot) \in \mathcal{V},$$

where $\mathcal{U}$ and $\mathcal{V}$ are feasible sets of strategies

$$\mathcal{U} = \{u(\cdot) \mid u(t) \in \mathbb{U}, t \in \mathcal{T}\}, \quad \mathcal{V} = \{v(\cdot) \mid v(t) \in \mathbb{V}, t \in \mathcal{T}\}.$$

Assume that the game in question begins in time instant $t = 0$ from a starting position $x_0 : x(0) = x_0$. The goal of player P consists of catching player E. And, conversely, player E must avoid being caught. This is a game of quality (with an outcome of boolean type), in which the pursuit is ceased in case of P catching E or running out of time $t = T$. The interception is considered successful if the distance between the players becomes less than some a priori fixed constant $\rho > 0$.

With $\mathcal{G} \subset \mathbb{X}$ we shall denote an object terminal set determining the interception condition. Therefore, the game ends if vector x enters set $\mathcal{G}$. Assume the moment of interception t^* is determined. Let set $\mathcal{G}$ be polyhedral:

$$\mathcal{G} = \{x \mid ||x|| \le \rho\},$$

where $||x||$ is a polyhedral norm of vector $x \in \mathbb{X}$.

As a measure of proximity of the game process to its ending we use the distance of the current position x to the origin, defined in the previously chosen polyhedral metric:

$$\gamma(x) = ||x||. \tag{5}$$

The game problem of interception consists in establishing a strategy — i.e. a means of players forming control laws u and v. Hence, player P approximates (minimizes its blunder) to a rendezvous point, and player E maximizes first player's blunder.

The above described class of linear discrete dynamic pursuit games has a polyhedral structure. That is why the authors name it *polyhedral* [14]. This class is also known as *polytopic games* [15]. Note that the first mention in the Russian language literature of the term "polyhedral game" is in a work of A. S. Belenskii [16], where it is used only for matrix games with polyhedral feasible sets of players' strategies.

The formulated discrete dynamic pursuit game can be interpreted as a problem of controlling an object under conditions of uncertainty [17]. Indeed, consider, for instance, the problem of stabilization an object into its equilibrium state (4). The controller's goal is to choose a control law that yields the minimization of the quality criterion (5) for the stabilization process. This choice is made regardless of the environment's aggressive behavior, assuming that it maximizes the disturbance of the controlled object. Therefore, the controller synthesis results in the need of using a game theory approach and formalization of the stabilization problem as a dynamic pursuit game.

B. Polyhedral pursuit strategy based on the principle of a guaranteed predicted blunder

As a basis of pursuit process control we use the idea of constructing a multi-step forecast (with fixed depth) on the future game development with subsequent estimation of the predicted cost — a terminal blunder. On every game step the Pursuer, using information on the current game state, generates a pursuit plan. It is a strict program based on a prognostic kinematic model of the game. The Pursuer takes into consideration the maximum possible counteraction of the Evader, and as a result constructs a control strategy oriented towards the worst game outcome — a maximum predicted terminal blunder.

The next five statements have critical effect on the proposed problem solution.

- The feasible sets of strategies available to the players are limited to "pure strategies" — i.e. strictly determined strategies ξ and η.
- The players' choice of control laws on every game step is based on a T-step plan (prediction) of the game's future development ($T \ge 1$). For every current time instant t the future point $t_h = t + T$ we call *horizon of planning* (*prediction*), the value T — *prediction depth*, and the time interval $[t, t_h]$ — *interval of planning* (*prediction*).

Let $\hat{x}(t+T \mid t)$ denotes a predicted game state. The corresponding value of measure (5)

$$\hat{\gamma} = ||\hat{x}(t+T \mid t)|| \tag{6}$$

we call a *predicted blunder*.

- The efficiency of both players' chosen control strategies is measured by the value of a predicted blunder (6).
- Player P on every step knows the current state $x(t)$ and, using the given prognostic model, develops a new pursuit plan (control strategy) in the form of a strict program.
- During the game process every player chooses a strategy that is always oriented towards the best possible future choice of its opponent. Hence, the players stick to the *principle of guaranteed result* (by Ju. B. Germeier) of predicted blunder. For instance, player P chooses the best possible strategy under condition that player E simultaneously chooses the worst one regarding P. This principle satisfies a more general *concept of preservation state* (by N. N. Krasovskii) in the means of measure (6). The latter guarantees that parameter (6) is not going to take worse values in subsequent steps.

As a basis for a method solving this pursuit problem we choose the following terminal prognostic construction. Fix $t \in \mathcal{T}$ and $x(t)$. Consider an auxiliary terminal game of interception:

1) The process of pursuit of players P and E is described by the following equation

$$\hat{x}(\theta+1 \mid t) = A\hat{x}(\theta \mid t) + \hat{u}(\theta \mid t) + \hat{v}(\theta \mid t), \tag{7}$$

where $\theta \in [t, t+T-1] \subset \mathbb{Z}_+$ is the current time, $\theta = t$ is the starting time, $\theta = t+T$ is the time instant of game's end.

2) Fix the initial game state $\hat{x}(t \mid t) = x(t)$.
3) The game cost is defined as the distance of the terminal state x from the origin, using a chosen polyhedral norm:

$$\gamma(t) = ||\hat{x}(t+T \mid t)||,$$

and is called a *terminal blunder*.

4) Players possess only the information of the initial state $x(t)$.
5) The feasible sets of strategies for both players are limited to the classes of control programs:

$$\hat{u}(\cdot) \in \mathcal{U}, \quad \hat{v}(\cdot) \in \mathcal{V},$$

where $\mathcal{U} = U^T$, $\mathcal{V} = V^T$ (Decart power of sets U and V).

Using (7) multiple times we find

$$\hat{x}(t+T \mid t) = A^T x + \sum_{\theta=t}^{t+T-1} A^{t+T-1-\theta}\hat{u}(\theta \mid t) + \sum_{\theta=t}^{t+T-1} A^{t+T-1-\theta}\hat{v}(\theta \mid t). \tag{8}$$

We will need the following sets

$$\mathcal{P} = A^T x + \sum_{\theta=0}^{T-1} A^{T-1-\theta} U \tag{9}$$

$$\mathcal{Q} = -\sum_{\theta=0}^{T-1} A^{T-1-\theta} V, \tag{10}$$

and vectors $y \in \mathcal{P}$ and $z \in \mathcal{Q}$ corresponding to control laws $\hat{u}(\cdot)$ and $\hat{v}(\cdot)$, such that holds

$$y = A^T x + \sum_{\theta=t}^{t+T-1} A^{t+T-1-\theta}\hat{u}(\theta \mid t),$$

$$z = -\sum_{\theta=t}^{t+T-1} A^{t+T-1-\theta}\hat{v}(\theta \mid t).$$

Then equation (8) can be replaced by

$$\hat{x}(\theta+1 \mid t) = y - z.$$

Therefore, sets (9) and (10) may be interpreted as *areas of reachability* for players P and E in the game space $\mathbb{X}$. Moreover, the game cost formulates as

$$\gamma = ||y - z|| = f(y - z).$$

Reaching the time instant t_h player P tries to move vector $\hat{x}$ closer to the origin, and, conversely, player E — as far as possible from the origin. Theses goals are achieved by the following strategies:

- *Minimax strategy for player P*, which has an expected guaranteed blunder f_P^* — the value of a polyhedral programming problem of minimax type

$$f_P^* = \min_{y \in \mathcal{P}} \max_{z \in \mathcal{Q}} f(y - z); \tag{11}$$

- *Maximin strategy for player E*, which has an expected guaranteed blunder f_E^*, with a value

$$f_E^* = \max_{z \in \mathcal{Q}} \min_{y \in \mathcal{P}} f(y - z). \tag{12}$$

Note that f_P^* is always not less than f_E^* ($f_P^* \geq f_E^*$).

As a result, the solution of the auxiliary terminal problem produces optimal control programs for both players $\hat{u}^*[\theta; t, x]$ and $\hat{v}^*[\theta; t, x]$, $t \leq \theta \leq t+T-1$. Now we can go back to the main problem, which game process is described by equation (4). For simplicity we choose the side of the Pursuer P, then E becomes our opponent. From the above assumptions it follows that player P knows the current game state $x(t)$. Now he has to solve the proposed auxiliary terminal pursuit game (model (7)) in order to construct a control program. Therefore, player P solves a minimax problem of an expected blunder. The obtained control sequence is used for selecting control law on the current step. It matches the first element of the optimal control program $\hat{u}^*[\theta; t, x]$ — i.e. $u[t] = \hat{u}^*[t; t, x]$. After reaching the next step in the game player P observes new game state $x(t)$ and solves a new auxiliary problem. Corrects his control program and uses only the first element of the new sequence $\hat{u}^*[\theta; t, x]$ as the current control law, and so on.

The proposed rule of constructing a rational control strategy for player P we call the *principle of a guaranteed predicted blunder*. This concept means that the Pursuer controls its motion in order to intercept the pursued object, with regard to uncertainty in the future movement of the latter. Let us outline the difference of our principle from the famous one of *extreme aiming*, also known as the rule of extreme choice of direction. It was formulated in 1963 by N. N. Krasovskii and successfully evolved in the game problems of intersecting motion trajectories [18]. Indeed, the Pursuer would choose a direction which, from the Pursuer's point of view, is going to be the worst future position of the Evader. Both concepts are of heuristic type. Although, we believe that the proposed principle of a guaranteed predicted blunder better meets the logic and practice of pursuit game problems.

We should note, that the increase of the prediction depth inevitably leads to increase of uncertainty about the planned control process development and algorithmically complicates computation of the optimal strategies. That is why one should choose the prediction depth adequate to the current game state. In particular, if for some horizon the Evader becomes reachable for the Pursuer ($\mathcal{Q} \subset \mathcal{P}$), then further increase of the depth is unnecessary. Hence, as the distance between the objects becomes shorter, one should decrease the prediction depth.

In conclusion, constructing control strategies by both players using the principle of a guaranteed predicted blunder leads to solving optimization problems (11) and (12), which belong to the class of of minimax and maximin polyhedral programming problems [4].

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