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Effects of a Physical Librations of the Moon Caused by a Liquid Core, and Determination of the Fourth Mode of a Free Libration[1]

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Abstract—An analytical theory of lunar physical librations based on its two-layer model consisting of a non-spherical solid mantle and ellipsoidal liquid core is developed. The Moon moves on a high-precision orbit in the gravitational field of the Earth and other celestial bodies. The defined fourth mode of a free libration is caused by the influence of the liquid core, with a long period of 205.7 yr, with amplitude $S = 0''0395$ and with an initial phase $\Pi_0 = -134°$ (for the initial epoch 2000.0). Estimates of dynamic (meridional) oblatenesses of a liquid core of the Moon have been estimated: $\varepsilon_D = 4.42 \times 10^{-4}$, $\mu_D = 2.83 \times 10^{-4}$ ($\varepsilon_D + \mu_D = 7.24 \times 10^{-4}$). These results have been obtained as a result of comparison of the developed analytical theory of physical librations of the Moon with the empirical theory of librations of the Moon constructed on the basis of laser observations.

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INTRODUCTION. STATEMENT OF PROBLEM

In recent years, important results have been obtained for the study of the internal structure of the Moon by the method of seismic tomography (Weber et al., 2011) and a highly accurate empirical theory of physical libration of the Moon was built based on extensive data on its laser observations (Rambaux and Williams, 2011). These works formed the basis for our study of the rotation of the Moon. For determination of the parameters of a ellipsoidal liquid core, we have used certain ratios of the moments of inertia of a core and the Moon, obtained in recent papers (Williams et al., 2010; 2011; 2012) on the basis of laser observations.

In the paper (Weber et al., 2011), on the basis of seismic data of the era of lunar *Apollo* missions to the Moon and using modern methods of analysis of seismic signals on the Moon (taking into account properties reflected and the transformed signals from a core), strong arguments were obtained in favor of the existence of a solid and a liquid core with a radius of 240 km and 330 km, respectively. In this article, we use these results to determine seismographic dynamic parameters of the core and mantle of the Moon in order to further study the effect of the liquid core of the Moon on its physical librations.

In this paper we consider the rotation of the Moon on the basis of its two-layer model (Ferrandiz and Barkin, 2000; 2003), consisting of a solid mantle and ellipsoidal liquid core with a perfect fluid, which performs a simple movement according to Poincaré (Poincaré, 1910; Lamb, 1947). A solid core in this paper is excluded from consideration.

Orbital motion of the Moon is described by the high-precision long-term numerical theory of DE/LE-406, on the base of which were built the required theory of the rotation of the Moon in the expansion of spherical functions of the coordinates of the Moon in the series on multiple Poisson arguments of the theory of orbital motion l_M, l_S, F and D (Kudryavtsev, 2007; Barkin et al., 2009). As a result, using the specified auxiliary expansions, expansions of a second harmonic of the force function of the gravitational potential in the variables of Andoyer–Poincaré (Barkin et al., 2009) were built. The limited space in the article doesn't allow a complete description of the constructed analytical theory of rotation of the Moon, therefore, here we provide only the description of the structure of the solution of a problem of forced and free librations of the Moon, and we concentrate attention on the determination and interpretation of a new mode of free libration of the Moon (the corresponding oscillations of the pole of its axis of rotation), caused by the influence of a liquid ellipsoidal core. As a result of comparison of the analytical theory of the physical librations of the Moon we con-

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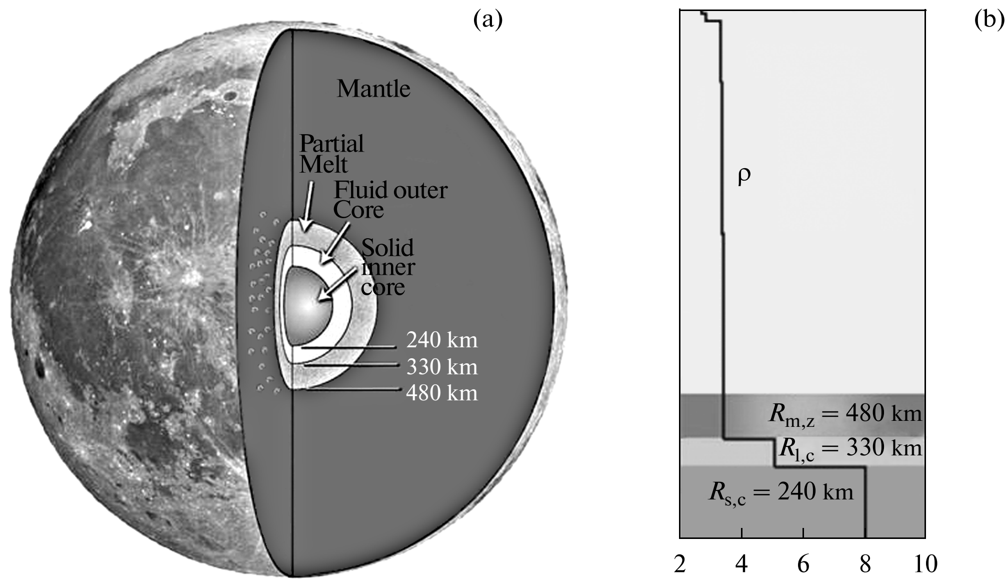

Fig. 1. The main shells of the Moon (a), their radii and graph of density ρ (b) (Weber et al., (2011).

structed, in particular its free librations, and the available empirical theory of rotation of the Moon, we determined the amplitude, initial phase, and period of the fourth mode of a free libration of the Moon. The main result here is that for the first time, we determined the period of free nutation of a core ($T_{\Pi} = 77757.032$ days), its amplitude and initial phase. On the basis of the specified value of the period is estimated the sum of two meridian compressions of a core $\varepsilon_D + \mu_D = 7.00 \times 10^{-4}$. For reasons of similarity and identity of relations of dynamic oblatenesses for the whole Moon and its core we have determined their individual values (Barkin et al., 2012).

For study of a rotary motion of the two-layer Moon it is necessary to determine first of all the main moments of inertia of the Moon (as full system) *A*, *B* and *C*, and values of the moments of inertia of its core and, respectively, dynamic compressions of a core. The specified parameters are the main ones in the problem and they define the forced and free librations of the Moon and its core in the gravitational field of the Earth and other celestial bodies. Estimates of the specified parameters of a core are executed.

MODEL OF THE MOON, ITS LIQUID CORE AND SOLID CORE

Main Parameters and Dynamic Characteristics

In the work (Weber et al., 2011), the shell model of the Moon (Fig. 1a) was built by seismic methods with solid and liquid cores with radii about 240 and 330 km. The existence of a partially molten zone around the liquid core with a radius of 480 km, or a spherical shell with radii of 330–480 km (Fig. 1a) were confirmed. The liquid core of the Moon (or liquid spherical shell with radii of 240–330 km) takes about ~60% of the volume of a core and makes the greatest contribution to the value of the polar moment of inertia of the core. Here we use the results of this work to determine the dynamic parameters of the core and mantle of the Moon to further investigate the effect of the liquid core of the Moon on physical librations.

Masses and moments of inertia of the Moon shells. In this model, the mean values of the density of the liquid core and solid core (or their homogeneous models) are: $\delta_{l,c} = 5.11$ g/cm^3 and $\delta_{s,c} = 8.04$ g/cm^3 (see Fig. 1b). Thus, the excess density of a solid core in comparison with density of a liquid core is $\Delta\delta_{s,l} = 2.93$ g/cm^3. In the future, we neglect the errors in determining the average density. For values of an average radius of a solid core and a liquid core we have the following estimates: $R_{l,c} = 330 \pm 20$ km, $R_{s,c} = 240 \pm 10$ km (Weber et al., 2011).

We emphasize that the model of a homogeneous solid core and its liquid shell are in good agreement with data of seismic observations (Fig. 1b). Therefore, a number of dynamic characteristics of the core (the polar moments of inertia, dynamic compressions, etc.), we construct, based on the corresponding homogeneous models in the form of homogeneous spheres or ellipsoids, using additional observational data, such as lunar laser ranging (Williams et al., 2011; 2012). We present the results of calculating the axial moment of inertia of the solid core, considered as a homogeneous sphere, and the liquid core as a homogeneous spherical layer.

For the solid core of the Moon, considered as a homogeneous spherical body with a radius $R_{s,c} = 240 \pm 10$ km, values of mass and polar moment of inertia are calculated by the formulas

$$m_{s,c} = \frac{4}{3}\pi\delta_{s,c}R_{s,c}^3, \quad C_{s,c} = \frac{8}{15}\pi\delta_{s,c}R_{s,c}^5 \tag{1}$$

and according to our estimates are

$$\begin{aligned} m_{s,c} &= (4.65 \pm 0.58) \times 10^{23} \text{ g}, \\ C_{s,c} &= (1.07 \pm 0.23) \times 10^{38} \text{ g cm}^2. \end{aligned} \tag{2}$$

Similarly, we determine the mass and moment of inertia of the liquid core, also considered as a spherical homogeneous layer of liquid, by formulas:

$$m_{l,c} = \frac{4}{3}\pi\delta_{l,c}\left(R_{l,c}^3 - R_{s,c}^3\right) = (4.73 \pm 1.78) \times 10^{23} \text{ g}, \tag{3}$$

$$\begin{aligned} C_{l,c} &= \frac{8}{15}\pi\delta_{l,c}\left(R_{l,c}^5 - R_{s,c}^5\right) \\ &= (2.67 \pm 0.81) \times 10^{38} \text{ g cm}^2. \end{aligned} \tag{4}$$

The result is that the total mass and the polar moment of inertia of the entire core including a liquid core and a solid core, will equal:

$$m_c = m_{s,c} + m_{l,c} = (9.38 \pm 1.61) \times 10^{23} \text{ kg}, \tag{5}$$

$$C_c = C_{s,c} + C_{l,c} = (3.74 \pm 1.04) \times 10^{38} \text{ g cm}^2. \tag{6}$$

It is important to note that value of the moment of inertia of a liquid core (4) makes about 71.3% of the moment of inertia of the entire core (6) (together with a solid core). It means that in the dynamics of a rotary motion of the Moon, the prevailing role is played by the liquid core. To a certain degree it justifies our choice of a two-layer model of the Moon for early studies of the dynamic role of a liquid core. However, in the subsequent works, we plan to consider features of dynamics of a three-layer model of the Moon consisting of a mantle, a liquid core and a solid core. Thus, for the considered model of a core its mass (5) is determined with a relative error of 17.2%, and the polar moment of inertia (6) is determined with a relative error of 27.8%. The average density of the homogeneous liquid core (spherical shell) is 6.233 g/cm^3. Thus, the mass and the polar moment of inertia of a core for two-layer and three-layer models of the Moon (Weber et al., 2011) coincide.

Dynamic parameter of influence of the liquid core to rotation of the Moon. To study the effects in the libration of the Moon, caused by a liquid core, an important role is played by the parameter L_c equal to the relation of axial polar moments of a core C_c and the whole Moon C. We will assume the known value of the moment of inertia of the Moon $C = 0.873486 \times 10^{42}$ g cm^2 (Araki et al., 2009). As a result, using the obtained value of the moment of inertia of the core (6) we obtain the value

$$L_c = C_c/C = (4.28 \pm 1.19) \times 10^{-4} \tag{7}$$
(with an accuracy of 27.8%).

Actually this estimate of a fundamental parameter of the Moon L_c derived on the basis of data from seismic studies of the Moon.

Parameter L_c is determined here most precisely in comparison with the previous estimates. Nevertheless, errors in determination of the dynamic parameter L_c are quite considerable. Therefore, in early studies of effects in rotation of the Moon, caused by a liquid core, we will be limited to consideration of a two-layer model.

Dynamic compressions of a core. In a rotary motion of the Moon we will study dynamic effects of a liquid core on the basis of Poincaré's model for a solid body with the ellipsoidal cavity filled with ideal liquid, on the basis of special forms of the equations of motion in variables of Andoyer and Poincaré, introduced in the previous works (Ferrandiz and Barkin, 2000; 2003). The ellipsoidal parameter of a liquid core is a predominating factor for studying the effects of a free nutation of the Moon due to the hydrodynamic influence of the liquid core. It can be characterized by semiaxes of an ellipsoidal cavity in which the liquid core is located, a_c, b_c and c_c. The axial moments of inertia A_c, B_c, C_c and parameters of inertia of Poincaré D_c, E_c, F_c are determined by formulas (Poincaré, 1910):

$$\begin{gathered} A_c = \frac{1}{5}m_c\left(b_c^2 + c_c^2\right), \quad B_c = \frac{1}{5}m_c\left(a_c^2 + c_c^2\right), \\ C_c = \frac{1}{5}m_c\left(a_c^2 + b_c^2\right), \\ D_c = \frac{2}{5}m_c a_c b_c, \quad E_c = \frac{2}{5}m_c a_c c_c, \quad F_c = \frac{2}{5}m_c b_c c_c, \end{gathered} \tag{8}$$

where m_c is the mass of the liquid core. We denote the coordinate axes so that for the moments of inertia and the semiaxes of the ellipsoidal core the following relations hold $C_c > B_c > A_c$, $a_c > b_c > c_c$, and introduce the dynamic compressions of a ellipsoidal core:

$$\varepsilon_D = 1 - A_c/C_c \quad \text{and} \quad \mu_D = 1 - B_c/C_c. \tag{9}$$

Seismographic methods currently do not allow us to identify and evaluate the geometric compression of a core and values of the semiaxes of the ellipsoidal cavity. However, research on the physical librations of the Moon based on perennial laser observations allow us to make it.

Estimates of dynamic compressions of the core based on the data of laser observations. One of the first estimates of the parameter $L_c = C_c/C = 6 \times 10^{-4}$ was obtained for a simple model of a homogeneous iron core with a radius of about 340 km and the compression of the core (axisymmetric model) was rated as 3×10^{-4}. In this paper, we obtained the concordant assessments of dynamic compressions of the core, including assessments on the basis of comparison of the developed analytical theory and the empirical theory based on laser observations (Rambaux and Williams, 2011).

In recent works (Williams et al., 2011; 2012), the first attempts to determine the dynamic compressions of a core from the results of the dynamic analysis of rotation of the Moon based on high-precision laser observations were made. In work (Williams et al., 2011), a similar assessment was obtained for the following combination entered in our work of the parameters of a core

$$L_c(\varepsilon_D + \mu_D)/2 = (2.3 \pm 0.8) \times 10^{-7} \quad \text{(error of 35\%)}. \tag{10}$$

Below we obtained an estimate for the sum of dynamic compressions of the core of the Moon by comparing our analytical theory of lunar physical librations (with liquid core) and the empirical theory of physical librations, constructed on the basis of perennial laser observations in a recent paper (Rambaux and Williams, 2011). On the basis of this value ($\varepsilon_D + \mu_D = 7.244 \times 10^{-4}$), obtained as a result of the analysis of librations of the Moon, from a ratio (10) we find an assessment for parameter

$$L_c = (6.57 \pm 2.29) \times 10^{-4} \text{ (error of 34.9\%)}. \tag{11}$$

In other work (Williams et al., 2012) another value of the ratio is given

$$L_c(\varepsilon_D + \mu_D)/2 = (1.6 \pm 0.7) \times 10^{-7} \quad \text{(error of 44\%)}. \tag{12}$$

In this work, as a result of study of the free librations of the Moon, on the basis of a known ratio between compressions and Poincaré's period of free nutation of a core (29), we have obtained an assessment $\varepsilon_D + \mu_D = 7.00 \times 10^{-4}$ for the sum of dynamic compressions of the core, to which the corresponding value of the parameter

$$L_c = (4.57 \pm 2.00) \times 10^{-4} \text{ (error of 44\%)}. \tag{13}$$

This value close to the value of this parameter (7) obtained above (by seismographic data), but with a little smaller error.

STRUCTURE OF THE SOLUTION OF THE PROBLEM OF THE PHYSICAL LIBRATIONS OF THE MOON AND MODES OF FREE LIBRATIONS

Space doesn't allow us to give a full statement of the work on the creation of the analytical theory of a rotary motion of the Moon with an ellipsoidal liquid core. Therefore, here we will concentrate attention on the analysis of free (resonant) libration of the Moon caused by a gravitational attraction of Earth and the hydrodynamic influence of a liquid core. Nevertheless, we will point to the previous general structure of the solution of a problem we solved concerning physical librations of the Moon:

$$\begin{gathered} \mathbf{Z}(t) = \mathbf{Z}_0 + \langle \mathbf{Z} \rangle + \tilde{\mathbf{Z}}(l_M, l_S, F, D, \ldots) \\ + \mathbf{Z}_{\text{res}}(P, Q, R, S, U_p, U_q, U_r, U_s, l_M, l_S, F, D, \ldots, t). \end{gathered} \tag{14}$$

In (14) we used the vector notation of several groups of variables used in the theory of rotation of the Moon. Among them, Andoyer variables, describing the rotational motions of the mantle and the core of the Moon (Ferrandiz and Barkin, 2000; Barkin et al., 2012):

$$\mathbf{Z} = (L, G, H, l, g, h; L_c, G_c, H_c, l_c, g_c, h_c). \tag{15}$$

Also classical variables of theories of physical librations of the Moon (Rambaux, Williams, 2011) are used:

$$\mathbf{Z} = (P_1, P_2, \tau, \rho, I\sigma) \tag{16}$$

and some other dynamic characteristics. In (16) τ, ρ and σ are librations in the longitude and an inclination, I is an average angle of inclination of the axis of rotation of the Moon relative to the normal to the plane of the ecliptic. In (14), arguments of the theory of orbital motion of the Moon are also listed: l_M, l_S, F, D, are the linear functions of time with certain frequencies. The geometric and dynamic sense of all the above variables is well known (Ferrandiz and Barkin, 2000; Gusev and Petrova, 2008; Barkin, et al., 2012).

Basic equations of rotational motion of the Moon with a liquid ellipsoidal core written in canonical form (Andoyer variables) and special methods developed for them of construction of quasi-periodic solutions of Hamiltonian systems under resonance conditions, study their linear neighborhood, and in the general case also the nonlinear neighborhood. These research methods were developed originally for the solid model of the Moon without the liquid core (Barkin, 1987; 1989), and in this paper they develop a more general model of a celestial body with an ellipsoidal liquid core. The method of small parameters which is introduced on the basis of the assumption of a small value of dynamic compression of the Moon, or on the basis of the assumption of an almost concentric distribution of its density is thus used. Generalization of the method relates primarily to the study of the perturbed motion of the poles of the Moon and the axes of rotation of the liquid core. At the first stage of construction of the theory, as in the case of a solid Moon, the main regularities in its rotation are studied. It is shown that the results for the solid model and the model of the Moon with the liquid core in respect of the librations of the Moon's mantle, are the same. However, new provisions on the nature of the unperturbed motion of the core under the Cassini laws of motion of the Moon were added (Ferrandiz and Barkin, 2003).

The first three terms of the solution (14) describe the intermediate conditionally periodic solution of the problem

$$\mathbf{Z}_p = \mathbf{Z}_0 + \langle \mathbf{Z} \rangle + \tilde{\mathbf{Z}}(l_\mathrm{M}, l_\mathrm{S}, F, D, \ldots) \quad (17)$$

and they do not actually contain the initial conditions of the problem. Here the solution is constructed primarily in Andoyer variables in integer powers of a small parameter μ, which characterizes the order of smallness of the dynamic compressions of the Moon, its core and mobility of Moon's orbital plane.

In (17) $\mathbf{Z}_0$ is the basic solution, describing the rotational motion by the Cassini laws. It is determined as the involvement of members of the first order with respect to dynamic compressions, corresponding to the second harmonic of the force function of the Newtonian interaction of the nonspherical Moon and the Earth. As a result we received dynamic study all the provisions of the laws of Cassini and formulated more complete and accurate positions, supplementing them (Barkin, 1987; 2011). The main result here is a theoretical determination of the constant angle of Cassini $\rho = \rho_0$ an unperturbed value of the angle between the normal to the plane of the ecliptic and the axis of rotation of the Moon (in motion by Cassini). Value of ρ_0 determined from the trigonometric equation (Barkin, 1987; Barkin et al., 2009):

$$\mathrm{B}\cos 2\rho_0 + \mathrm{A}\sin 2\rho_0 + \mathrm{Y}\cos\rho_0 + \mathrm{U}\sin\rho_0 = 0, \quad (18)$$

whose coefficients (and, of course, the value of the angle ρ_0) depend on the main parameters of selenopotential C_{20}, C_{22} and parameters of the perturbed orbital motion of the Moon (in particular on the rate of precession of the lunar orbit n_Ω). Here we omit the detailed recording and analysis of the equation (18). But note that this equation is a necessary condition for the existence of quasi-periodic solutions (17), which describe the forced librations of the Moon.

Based on the parameters of a modern model of the gravitational field of the Moon, built as a result of the Japanese space mission *Selena I* (Matsumoto et al., 2010), based on equation (16) the value $\rho_0 = 1°32'37''9 = 5557''94$ (Barkin et al., 2012) is found. This theoretical value agrees well with the related value of the angle of Cassini $\rho = 5553''60$, obtained in the empirical theory from a long series of the Moon laser observations (Rambaux and Williams, 2011).

The motion of the Moon with the liquid core according to the laws of Cassini is determined by the conditions of existence of quasi-periodic solutions of the equations of motion of the Moon (from the equations of the first approximation for the canonical variables Andoyer–Poincaré) (Barkin et al., 2012). In the unperturbed motion the Moon with the liquid core makes axial solid-body rotation with constant angular velocity n_F, representing the frequency of the argument of the orbital motion of the Moon F.

More difficult is the problem of constructing analytic expressions for the constant components of variables $\langle \mathbf{Z} \rangle$ conditionally—periodic solutions $\mathbf{Z}_p$ (17). These permanent additions to the generating values of variables $\mathbf{Z}_0$, according to the method of developing, determined by analyzing the equations of the second and higher-order approximations, which are sequentially addressed in the construction of conditionally periodic solutions, in which $\tilde{\mathbf{Z}}(l_\mathrm{M}, l_\mathrm{S}, F, D, \ldots)$ its purely conditionally-periodic component. Perturbations $\tilde{\mathbf{Z}}(l_\mathrm{M}, l_\mathrm{S}, F, D, \ldots)$ is the sum of trigonometric terms with arguments that are linear combinations of the classical arguments of the theory of orbital motion of the Moon. This important part of constructing a theory of the Moon's rotation, we plan to devote a separate article with the results of the identification and tabulation of the amplitudes, periods and initial phases of forced physical librations in Andoyer variables, variables in the classical theory of the physical librations of the Moon, as well as for the components of the angular velocity of rotation of the Moon.

In this paper, we focus on the fourth term of the solution (14)

$$\mathbf{Z}_\mathrm{res}(P, Q, R, S, U_p, U_q, U_r, U_s, l_\mathrm{M}, l_\mathrm{S}, F, D, \ldots, t), \quad (19)$$

which describes free and resonant librations of the Moon. Here, P, Q, R, S—constant amplitudes and U_p, U_q, U_r, U_s—arguments of free librations Moon which are linear functions of time:

$$U_p = n_p t + U_p^{(0)}, \quad U_q = n_q t + U_q^{(0)}, \quad U_r = n_r t + U_r^{(0)}, \quad U_s = n_s t + U_s^{(0)}, \quad (20)$$

with these constant frequencies n_p, n_q, n_r and n_s. $U_p^{(0)}$, $U_q^{(0)}$, $U_r^{(0)}$ and $U_s^{(0)}$—initial values of the arguments—arguments phase (for the epoch 2000.0).

Thus, the solution of the problem of the physical librations of the Moon (14)–(20) (for the considered two-layer model of the Poincaré) contains 8 initial conditions. They represent the amplitude and phase of the free libration in longitude $(P, U_p^{(0)})$, in inclination $(R, U_r^{(0)})$, in motion of poles $(Q, U_q^{(0)})$ and in a free nutation of a core $(S, U_s^{(0)})$. The first 6 of the initial conditions of the problem of the physical libration of the Moon have been identified on the basis of the lunar laser ranging, for the first time in the work of Calame (1976). More precisely, these six initial conditions were identified in the current study (Rambaux and Williams, 2011). Despite the fact that this study was determined from observations of a number of new librations with certain periods, but of unknown origin, the authors were unable to determine the parameters of the fourth mode of the free libration $(S, U_s^{(0)})$. Such unidentified librations the theory (Rambaux and Williams, 2011) contains about fifty to five classical variables $\mathbf{Z} = (P_1, P_2, \tau, \rho, I\sigma)$ (16). In this paper, we first identify the fourth mode of the free librations of the Moon in the solution (14)–(20) and give a first evaluation of its period, amplitude and initial phase.

Due to the resonant nature of the Moon's motion, analytical solution of the variational equations (19) $\mathbf{Z}_{\text{res}}(P, Q, R, S, U_p, U_q, U_r, U_s, l_M, l_S, F, D, \ldots, t)$ we searched for integer and fractional powers of a small parameter (in powers $\sqrt{\mu}$). Such a method for solving the variational equations was previously applied in the theory of rotation of a solid non-spherical Moon without the liquid core (Barkin, 1987), and later in the study of the Moon's rotation with a liquid core (Ferrandiz and Barkin, 2003; Barkin et al., 2012). A detailed description of the analytical solutions (14)–(20) will be given in our subsequent work. Here we will focus on describing the effects of the free movement of the pole axis of rotation of the Moon, due to its ellipsoidal liquid core.

FREE LIBRATIONS OF THE MOON

Free librations of the Moon appear in all variables $\mathbf{Z}$ (14) briefly described above, in particular in the variables Andoyer. According to our solution, they represent the solution of equations in variations, describing the rotation of the Moon in neighborhood of its intermediate conditionally-periodic solution. We point out the structure of this solution in the canonical variables of Andoyer (Barkin et al., 2012):

$$\begin{aligned}\delta\mathbf{z} = {} & P\left(\mathbf{z}^{(p)}\cos U_p - \mathbf{z}^{(p)*}\sin U_p\right) \\ & + Q\left(\mathbf{z}^{(q)}\cos U_q - \mathbf{z}^{(q)*}\sin U_q\right) \\ & + R\left(\mathbf{z}^{(r)}\cos U_r - \mathbf{z}^{(r)*}\sin U_r\right) \\ & + S\left(\mathbf{z}^{(s)}\cos U_s - \mathbf{z}^{(s)*}\sin U_s\right).\end{aligned} \quad (21)$$

Here, $\mathbf{z} = (l, g, h, l_c;\ L, G, H, L_c)$—the vector of the canonical variables of Andoyer describing the rotational motion of the Moon. The basis of the analytical studies is special canonical equations of rotational motion of a rigid body (mantle) with ellipsoidal cavity filled with an ideal fluid, doing simple motion on Poincaré (Ferrandiz and Barkin, 2000). As well as methods for constructing quasi-periodic solutions of Hamiltonian systems containing a small parameter, and methods of analysis of its linear and nonlinear neighborhoods (Barkin, 1987; Barkin and Ferrandiz, 2003; Barkin et al., 2012).

In solving (21) P, Q, R, S—amplitudes of the free librations. Arguments U_p, U_q, U_r and U_s are determined by the formulas (20), where $U_p^{(0)}$, $U_q^{(0)}$, $U_r^{(0)}$, $U_s^{(0)}$—initial values of the arguments of free librations (at epoch 2000.0). Frequencies of the free librations of the Moon p, q, r and s are determined by solving of the corresponding characteristic equation. They are purely imaginary:

$$p = \pm ip_0,\quad q = \pm iq_0,\quad r = \pm ir_0,\quad s = \pm is_0, \quad (22)$$

where p_0, q_0, r_0 and s_0—the real frequency of free oscillations, which correspond to periods of oscillations

$$T_p = \frac{2\pi}{p_0},\quad T_q = \frac{2\pi}{q_0},\quad T_r = \frac{2\pi}{r_0},\quad T_s = \frac{2\pi}{s_0}. \quad (23)$$

These frequencies and the corresponding periods lend themselves to precise analytical description and their ratings are not required observations of librations of the Moon, but it requires the exact values of the parameters of the dynamic structure, its gravitational field and the characteristics of the ellipsoidal core. And on the contrary, values of amplitudes of free librations P, Q, R, S and their initial phases $U_p^{(0)}$, $U_q^{(0)}$, $U_r^{(0)}$, $U_s^{(0)}$ can only be determined on the basis of the data of observations of the rotational motion of the Moon. Up to the present time, from these four modes according to laser observations were determined only the first three. The very first determination of the amplitudes and phases of the free librations of the Moon by laser measurements of distances to the reflectors mounted on the surface of the Moon was made in the work of Calame (1976). Modern high-precision determination of the amplitudes and phases of three modes of free librations of the Moon was made in a recent paper (Rambaux and Williams, 2011). The authors used a long series of laser observations of the Moon (for a period of about 40 years).

In this paper, we omit the detailed derivation of the formulas for the frequency and period of free librations of the Moon (all four modes) and the formulas for the solution of variational equations in Andoyer's variables, and the final results for subsequent analysis and the transition to the variations of the classic variables of the theory of physical librations of the Moon $\mathbf{Z} = (P_1, P_2, \tau, \rho, I\sigma)$ (16). The most important issue here is the question of the determination of the fourth mode of the free librations of the Moon caused by the liquid core on the base of empirical data (Rambaux and Williams, 2011).

The main resonance effects in the rotational motion of the Moon and their interpretation. In this paper, we study the fundamental resonance effects in the rotational motion of the Moon in a neighborhood of a intermediate conditionally-periodic motion (17). The Moon has a core with a small size (figure). Because of this feature the general characteristic equation in variations in a first approach is split into two equations. The first equation defines three modes of free librations in longitude, in inclination and free oscillations of the pole with frequencies p_0, q_0, r_0. This equation we studied in the theory of physical librations of the solid Moon model. In particular we have been evaluated three periods of free librations of the Moon, corre-

Table 1. Four modes of free librations of the Moon

Amplitudes	Arguments	Periods, days	Phases, deg	Modes
$P = 1''735$	$U_p(t) = pt + U_p^{(0)}$	1057.13	$U_p^{(0)} = 207.01°$	Free libration in longitude with period 2.99 yr
$Q = 3''3072$	$U_q(t) = qt + U_q^{(0)}$	27257.27	$U_q^{(0)} = 161.60°$	Free oscillations of the pole with period 74.3 yr
$R = 1''1881$	$U_r(t) = rt + U_r^{(0)}$	8822.88	$U_r^{(0)} = 160.81°$	Free oscillations of the angular momentum vector in space with a period 24.3 yr
$S = 0''0160$	$U_s(t) = st + U_s^{(0)}$	27.312	$U_s^{(0)} = 39°$	Quasi-diurnal variation of the pole of the Moon with the period 27.312 days

sponding to these frequencies (Barkin, 1987; Barkin et al., 2012). The second characteristic equation allows to determine the frequency of the fourth mode of free oscillations of the pole of the Moon, caused by a liquid ellipsoidal core of the Moon. In case of a small core for this frequency we received analytical expression in terms of key parameters of considered model of the Moon:

$$s_0 = n_F\sqrt{\frac{1}{\Delta_1\Delta_3}\left(ACE_c^2 - 2BE_cF_cD_c + \frac{B^2}{AC}F_c^2D_c^2\right)}, \quad (24)$$
$$\Delta_1 = A_cA - F_c^2, \quad \Delta_3 = C_cC - D_c^2.$$

Saving in expressions (24) only the terms of the first degree of smallness relative to dynamic compressions of the core (9), for the frequency (24), we obtain a known simplified expression

$$s_0 = n_F + n_F(\varepsilon_D + \mu_D)/2. \quad (25)$$

Corresponding to the frequencies p_0, q_0, r_0 and (25) periods (23) are determined by the following formulas:

$$T_p = \frac{n_F}{n_0}\frac{T_F}{\sqrt{f_0^{(g,g)}}}, \quad (26)$$

$$T_q = \frac{n_F^2}{n_0^2}\frac{T_F}{\sqrt{f_0^{(\theta,\theta)}f_0^{(l,l)}}}, \quad (27)$$

$$T_r = \frac{n_F^2}{n_0^2}\frac{T_F\sqrt{f_0^{(g,g)}}}{\sqrt{f_0^{(H,H)}\left[f_0^{(g,g)}f_0^{(h,h)} - \left(f_0^{(g,h)}\right)^2\right]}}, \quad (28)$$

$$T_s = T_F - T_F\frac{2}{\varepsilon_D + \mu_D}. \quad (29)$$

In formulas (26)–(29) T_F draconic period of the lunar orbital motion $T_F = 2\pi/n_F = 27.21222$ days. Here, the frequency $n_0 = \sqrt{fm_\oplus/a^3}$ (f is the gravitational constant, $m_\oplus$ is the mass of the Earth, a is an average semi-major axis of the lunar orbit, accepted in our theory by the equal a = 383397772.5 m). The numerical values of these frequencies are

$$n_F = 17395266.10''/\text{yr}, \quad n_0 = 17243514.58''/\text{yr}. \quad (30)$$

For the assumed values of the dynamic compressions of the Moon according to the formula (29) we obtain the following theoretical value of the period of free libration, due to the influence of the liquid core T_s = 27.1976 days. Dimensionless coefficients $f_0^{(l,l)}$, $f_0^{(L,L)}$, ..., $f_0^{(g,g)}$ in the formulas (26)–(28) are the normalized values of the respective second partial derivatives of the averaged perturbing Hamiltonian of the problem $\langle F_1\rangle$. They are defined rather cumbersome formulas were first computed in (Barkin, 1987) for a model of the gravitational field of those times (Gusev and Petrova, 2008). In this paper, for these coefficients were obtained by the numerical values:

$$f_0^{(l,l)} = 2506.765\times10^{-6}, \quad f_0^{(g,g)} = 672.2999\times10^{-6},$$
$$f_0^{(H,H)} = -4.331321, \quad f_0^{(L,L)} = 410.4448\times10^{-6},$$
$$f_0^{(g,h)} = 673.1132\times10^{-6}, \quad f_0^{(h,h)} = 671.6517\times10^{-6},$$
$$f_0^{(\theta,\theta)} = 410.4448\times10^{-6}, \quad (31)$$

but already for modern model of a gravitational field of the Moon constructed by authors of work (Matsumoto et al., 2010).

Now, according to the formulas (26)–(31) for the accepted values of the parameters of the model we find the values of the four periods of free librations of the Moon with liquid ellipsoidal core. They are shown in Table 1 in comparison with the corresponding periods of the empirical (laser) theory of the librations of the Moon (Rambaux and Williams, 2011).

To determine the amplitudes P, Q, R, S and initial phases $U_p^{(0)}$, $U_q^{(0)}$, $U_r^{(0)}$, $U_s^{(0)}$ of free librations we will use the empirical theory of physical librations of the Moon (Rambaux and Williams, 2011). For this purpose we will identify designations of the main arguments of

Table 2. Free and resonant librations of the Moon. Amplitudes, periods and trigonometric arguments, variations of the corresponding classical variables

N	Variables	Theory analytical	Theory empirical	Trigonometric terms	Periods, days
1	δP_1	$-3''3060$	$-3''306$	$\sin W$	27257.273
2	δP_1	$-0''0320$	$-0''032$	$\sin(V-F)$	27.296
3	δP_1	$0''0250$	$0''025$	$\sin(U-F)$	27.932
4	δP_1	$0''0222$	–	$\sin(U+F)$	26.529
5	δP_1	$-0''00004$	–	$\sin(V+F)$	27.129
6	δP_1	$-0''001$	$0''000$	$\cos\Theta$	27.312
7	δP_1	$-0''016$	$-0''016$	$\sin\Theta$	27.312
8	δP_2	$8''1830$	$8''183$	$\cos W$	27257.273
9	δP_2	$0''0320$	$0''032$	$\cos(V-F)$	27.296
10	δP_2	$-0''0250$	$-0''025$	$\cos(U-F)$	27.932
11	δP_2	$0''0222$	$0''022$	$\cos(U+F)$	26.529
12	δP_2	$-0''00004$	–	$\cos(V+F)$	27.129
13	δP_2	$-0''016$	$-0''016$	$\cos\Theta$	27.312
14	δP_2	$0''001$	$0''002$	$\sin\Theta$	27.312
15	$\delta\tau_K$	$1''7570$	$1''735$	$\sin U$	1056.210
16	$\delta\tau_K$	$0''0774$	$0''077$	$\sin(W+F)$	27.185
17	$\delta\tau_K$	$-0''0328$	$-0''032$	$\sin(W-F)$	27.239
18	$\delta\tau_K$	$-0''0014$	–	$\sin V$	8822.883
19	$\delta\tau_K$	$-0''001$	–	$\cos\Xi$	7449.890
20	$\delta\rho_K$	$5''7402$	$5''753$	$\cos(W+F)$	27.185
21	$\delta\rho_K$	$2''4330$	$2''437$	$\cos(W-F)$	27.239
22	$\delta\rho_K$	$0''0320$	$0''029$	$\cos V$	8822.883
23	$\delta\rho_K$	$-0''0026$	$-0''003$	$\cos U$	1056.210
24	$\delta\rho_K$	$-0''007$	$-0''013$	$\cos\Xi$	7449.890
25	$\delta\rho_K$	$0''049$	$0''052$	$\sin\Xi$	7449.89
26	$I\delta\sigma_K$	$5''7402$	$5''758$	$\sin(W+F)$	27.185
27	$I\delta\sigma_K$	$-2''4330$	$-2''443$	$\sin(W-F)$	27.239
28	$I\delta\sigma_K$	$-0''0320$	$0''033$	$\sin V$	8822.883
29	$I\delta\sigma_K$	$-0''049$	$-0''045$	$\cos\Xi$	7449.890
30	$I\delta\sigma_K$	$-0''007$	$-0''002$	$\sin\Xi$	7449.890

free librations from this work U, V and W with our designations:

$$U_p = U + \pi/2, \quad U_q = W - \pi/2, \quad U_r = V + \pi/2. \quad (32)$$

The same ratios are obtained for the initial phases of these argument:

$$U_p^{(0)} = U_0 + \pi/2, \quad U_q^{(0)} = W_0 - \pi/2, \quad U_r^{(0)} = V_0 + \pi/2.$$

In this study, we determined the initial values of all phases: $U_p^{(0)}$, $U_q^{(0)}$, $U_r^{(0)}$ and $U_s^{(0)}$. They are listed in Table 1. And also identified four values of the amplitudes of the main free librations (see below), frequencies (and the periods) of free oscillations.

For convenience of comparison of free librations (amplitudes, phases and the periods) in analytical and empirical theories in our solution in Andoyer's variables we will write down free librations in the form of (21), using for resonant arguments of their expression (32) through arguments U, V and W, used in theory based on laser observations.

As a result of the solution of the equations in variations by means of a method of small parameter we constructed the approximate solution of the problem of free librations in Andoyer's variables. Note that in the considered formulation of the problem of a rotation of the core model of the Moon mapped to a simple motion of an ideal liquid according to Poincaré (Lamb, 1947). The problem of the physical librations of the Moon was reduced to the solution of the canonical system of equations of the 8th order.

In this work, at the beginning the solution of the equations in variations was obtained in Andoyer's variables $\mathbf{z} = (l, g, h, l_c; L, G, H, L_c)$ and then was converted to the known classical variables $\mathbf{Z} = (P_1, P_2, \tau, \rho, I\sigma)$ (Rambaux and Williams, 2011). Also Andoyer's angular variables were used θ and ρ, for which $L = G\cos\theta$, $H = G\cos\rho$ and similar variables for motion of a liquid core: θ_c and ρ_c, for which $L_c = G_c\cos\theta_c$. Quite capacious calculations for the specified constructions we will lower for brevity statements and we will give final results on studying of free librations of the Moon, including determination of entry conditions of a task on the basis of their comparison with the empirical theory (Rambaux and Williams, 2011), constructed on the basis of laser observations. So for variations $\delta\mathbf{z} = (\delta l, \delta g, \delta h, \delta\omega, \delta\theta, \delta\rho; \delta l_c, \delta\theta_c)$ the solution of the variational equations in the vicinity of motion by Cassini's laws is represented in the following compact form:

$$\begin{aligned} &\delta\omega/n_F = \Lambda\cos U, && \delta g = E\sin U + \Gamma\sin V, \\ &\delta\theta = T\cos W - T_s\sin U_s, && \delta l = -L\sin W + Q_s\cos U_s, \\ &\delta\rho = -M\cos U + N\cos V, && \delta h = H\sin V, \\ &\delta\theta_c = -T_s^{(c)}\sin U_s, && \delta l_c = Q_s^{(c)}\cos U_s, \end{aligned} \quad (33)$$

Table 3. Unexplained librations with significant amplitudes (in classical variables of the theory of physical librations of the Moon)

Variables	Status	Periods, days	C, arcsec	S, arcsec
$I\sigma_K$	+Un	7481.531	−0.045	−0.002
ρ_K	+Un	7468.388	−0.013	0.052
P_1^K	+Un	27.312	0.000	−0.016
P_2^K	+Un	27.312	−0.016	0.002

where $\delta\omega = \delta G/B$ is a variation of the modulus of the angular velocity vector. In (33) we used a new notation for the amplitudes of the free librations described in Andoyer's variables:

$$\Lambda, T, T_s, M, N; \quad L, Q_s, H, E, \Gamma; \quad Q_s^{(c)} T_s^{(c)}. \quad (34)$$

For the amplitudes (34) were obtained analytical expressions, depending on the parameters of the model of the Moon and its perturbed orbital motion. These amplitudes are proportional to the amplitude of the free librations of the Moon: P, Q, R and S of all considered four modes of the librations. To carry out a comparison of our theory with the empirical theory (Rambaux and Williams, 2011) (to determine the initial conditions of the problem) it is necessary to transform the solution in Andoyer's variables (33), (34) to classical variables of the theory of a physical librations of the Moon (16).

Free librations in classical variables of the theory of rotation of the Moon. If the solution of a problem is constructed in Andoyer's variables, similar expressions can be obtained in classical variables of the theory of physical librations and for components of the angular velocity (of the Moon and its core). Indeed, between classical variables of the theory of physical librations and Andoyer's variables there are simple geometrical relations which directly follow from expressions of direction cosines of the principal axes of inertia of the Moon in the principal ecliptic reference system (Barkin, 1987).

At the first stage in our theory, Andoyer's variables were used and then the solution of the variational equations was constructed in classical variables (16). In the linear approximation these geometrical relations in particular allow us to express variations of classical variables $\mathbf{Z} = (P_1, P_2, \tau, \rho, I\sigma)$ through the corresponding variations of Andoyer variables δl, $\delta\theta$, δg,

δh and $\delta\rho$ (in a vicinity of motion of the Moon according to Cassini) (Barkin, 1987):

$$\delta P_1^{\mathrm{K}} = -\sin\rho\cos F\delta g - \cos\rho\delta l - \cos\rho\sin F\delta\rho,$$
$$\delta P_2^{\mathrm{K}} = \sin\rho\sin F\delta g - \cos\rho\delta\theta - \cos\rho\cos F\delta\rho,$$
$$\delta\tau_{\mathrm{K}} = \delta g + \delta h + \tan\frac{\rho}{2}(\sin F\delta\theta - \cos F\delta l), \quad (35)$$
$$\delta\rho_{\mathrm{K}} = \delta\rho + \sin F\delta l + \cos F\delta\theta,$$
$$I\delta\sigma_{\mathrm{K}} = -\cos F\delta l + \sin F\delta\theta + \sin\rho\delta h.$$

Here, the index $^{\mathrm{K}}$ (or $_{\mathrm{K}}$) is also attributed to classical variables. Substituting the solution for free librations in Andoyer's variables (33), (34) in formulas (35), we will obtain variations for 5 classical variables:

$$\delta P_1^{\mathrm{K}} = Q\cos\rho\sin W + Q_s\cos\rho\cos U_s$$
$$+ \frac{1}{2}(M\cos\rho - P\sin\rho)\sin(U + F)$$
$$- \frac{1}{2}(M\cos\rho + P\sin\rho)\sin(U - F) \quad (36)$$
$$+ \frac{1}{2}(\Gamma\sin\rho - N\cos\rho)\sin(V + F)$$
$$+ \frac{1}{2}(\Gamma\sin\rho + N\cos\rho)\sin(F - V),$$
$$\delta P_2^{\mathrm{K}} = -T\cos\rho\cos W - T_s\cos\rho\sin U_s$$
$$+ \frac{1}{2}(M\cos\rho - P\sin\rho)\cos(U + F)$$
$$+ \frac{1}{2}(M\cos\rho + P\sin\rho)\cos(U - F)$$
$$+ \frac{1}{2}(\Gamma\sin\rho - N\cos\rho)\cos(V + F)$$
$$- \frac{1}{2}(\Gamma\sin\rho + N\cos\rho)\cos(V - F),$$
$$\delta\tau_{\mathrm{K}} = P\sin U + (R - \Gamma)\sin V$$
$$+ \frac{1}{2}\tan\frac{\rho}{2}[(T + Q)\sin(F + W) + (Q - T)\sin(W - F)]$$
$$+ \frac{1}{2}\tan\frac{\rho}{2}[(T_s - Q_s)\cos(U_s + F) - (T_s + Q_s)\cos(U_s - F)],$$
$$\delta\rho_{\mathrm{K}} = -M\cos U + N\cos V$$
$$+ \frac{1}{2}(T - Q)\cos(W - F) + \frac{1}{2}(T + Q)\cos(W + F)$$
$$- \frac{1}{2}(T_s - Q_s)\sin(U_s + F) - \frac{1}{2}(T_s + Q_s)\sin(U_s - F),$$
$$I\delta\sigma_{\mathrm{K}} = \sin\rho R\sin V$$
$$+ \frac{1}{2}(T + Q)\sin(W + F) + \frac{1}{2}(-T + Q)\sin(W - F)$$
$$+ \frac{1}{2}(T_s - Q_s)\cos(U_s + F) - \frac{1}{2}(T_s + Q_s)\cos(U_s - F).$$

We recall that constant values of amplitudes of the free librations of the Moon (for all four modes of oscillations) P, Q, R and S appear as a factors in the expression of the amplitudes of free librations in formulas (36). Arguments U, V, W and U_s are the resonant arguments of free librations of the Moon. For the first three of them we will accept the same notation, as in the empirical theory (Rambaux and Williams, 2011) (see Table 1).

We will determine constant amplitudes and initial phases of arguments from a comparison of the analytical solution of a problem of free librations of the Moon (36) and tables of librations (free and in the same variables) for the empirical theory (Rambaux and Williams, 2011). As a result of the realization of this procedure, we determine amplitudes and initial phases of the first three modes of free librations of the Moon (in longitude, in an inclination and in pole oscillations). These values are given in Table 1 (in arcseconds): $P = 1''7570$, $Q = 3''3072$, $R = 1''1881$,

Taking into account representations (24)–(30) in formulas (23) in particular we obtain the corresponding values of frequencies (1 unit = degree/day):

$$p_0 = \frac{360°}{1056.13}, \quad q_0 = \frac{360°}{27257.27}, \quad r_0 = \frac{360°}{8822.88}. \quad (37)$$

Thus, at this stage we have identified the main free libration of the Moon which appears in variations of classical variables. The Julian date here is taken as an initial time point $t_0 = 2451545.0$, as in the work (Rambaux and Williams, 2011).

If the terms of the free librations of the Moon for the first three modes are quite clearly discernible in the empirical theory, then with the identification of the fourth mode of the free libration of the Moon, due to the influence of the liquid core, the situation is much more complicated.

As a result of comparison of amplitudes of librations in the solution (36) with their values obtained in the empirical theory, first we calculate values of constant factors

$$\Lambda = 0''04516, \quad T = 8''1732, \quad M = 0''00263646, \quad N = 0''03195, \quad \Gamma = 1''1895 \quad (38)$$

and, respectively, we find the initial values of the amplitudes of free librations of the first three modes (listed in Table 1). With values (38) and formulas (36) we find amplitudes of 30 terms of free librations of the Moon. They are listed in the Table 2 in comparison with similar amplitudes of the theory (Rambaux and Williams, 2011). Table 2 shows a good agreement with the analytical theory developed here (Barkin, 1987; Barkin et al., 2012) and the empirical theory constructed on laser observations (Rambaux and Williams, 2011).

In Table 2 we used auxiliary arguments

$$\Xi = (n_\Omega + n_\Pi)t, \quad \Sigma = (n_F + n_\Omega + n_\Pi)t. \quad (39)$$

These arguments are used for the description of free librations of the Moon caused by the influence of an

Table 4. The main librations of the Moon caused by a liquid ellipsoidal core

Variables method	Θ	Period, days	Amplitude $\cos\Theta$, arcsec	Amplitude $\sin\Theta$, arcsec
P_1 observations	Un	27.312	0.000	–0.016
P_1 theory	$F+\Omega+\Pi$	27.312	–0.001	–0.016
P_2 observations	Un	27.312	–0.016	0.002
P_2 theory	$F+\Omega+\Pi$	27.312	–0.016	0.001
ρ observations	Un	7468.39	–0.013	0.052
ρ theory	$\Omega+\Pi$	7449.89	–0.007	0.049
$I\sigma$ observations	Un	7481.53	–0.045	–0.002
$I\sigma$ theory	$\Omega+\Pi$	7449.89	–0.049	–0.007
τ observations	Un	–	–	–
τ theory	$\Omega+\Pi$	7449.89	–0.001	–

ellipsoidal liquid core (in Table 2 these librations are specified in lines with numbers 6, 7; 13, 14; 19; 24, 25; 29, 30).

Values of amplitudes and the periods are given in this table for 30 free variations of the main variables of the theory of librations of the Moon according to formulas of the constructed solution (36) in comparison with similar characteristics of the empirical theory of the rotation of the Moon (Rambaux and Williams, 2011). Here, amplitudes and the periods of free librations for projections of angular velocity vector of rotation of the Moon to its principal axes of inertia are specified. Our solution of a problem of free librations of the Moon along with classical members contains some additional members, which as shown by the numerical estimates given in Tables 3 and 4, are quite significant and have to be considered in light of laser ranging observations.

DETERMINATION OF THE FOURTH MODE OF FREE LIBRATIONS. DUE TO THE LIQUID CORE

The periods and amplitudes of modes of free librations which were determined are given in Table 1. The DE421 model of integration included a liquid core with the flattened core–mantle boundary. It adds the fourth mode of a possible free mode which may be called a retrograde precession of a pole in space, usually called a free nutation of a core by analogy with the Earth's rotation. This mode will mainly influence the liquid core, but there will also be a small reaction to the mantle librations.

According to the DE421 theory, flattening at the core boundary causes a spatial oscillation of the Moon with a period of about 197 years. The period of a free nutation of the core depends on the orbital period and core flattening. This period expressed in days is estimated by a formula (Rambaux and Williams, 2011):

$$P_{FCN} \sim 27.3\frac{2}{f_c},$$

where f_c is a flattening on the DE421 model equal to 3.8×10^{-4}. In expressions of classical variables in the theory of librations of the Moon, according to the theory of DE421, the ρ_K and $I\sigma_K$ integration has to show that the period of progressive motion is about 7500 days. The libratory term with the period of 7367 days was identified with the forced librations predicted by Eckhart (Rambaux and Williams, 2011).

Two trigonometrical terms of a free nutation of a core with the period in 7481 days for variable $I\sigma_K$ and with a period of 7468 days, for a variable ρ_K with amplitudes of 0.045 and 0.054 angular seconds, respectively, are candidates. Equivalent terms in variables P_1 and P_2 are terms with periods 27.312 days and with an amplitude of 0.016 arcsec.

Thus, the amplitudes in the two representations are different from each other.

Authors (Rambaux and Williams, 2011) showed as a result of careful analysis that terms that candidates for a role of free librations from a core with the periods of 7481 and 7468 days strongly (0.95) correlate with above-mentioned variations with a period of 7367 days. So the beat period between these periods is about 1000 years, which corresponds to the period covered by an ephemeris. Thus, it is thus noted that amplitudes of members of candidates for a role of members of free nutation in variables $I\sigma$ and ρ, apparently, are considerably overestimated because of strong correlation, and the amplitude of similar members in two other variables P_1 and P_2 look more realistic. These numerical results are applicable to the DE421 theory, but real values of flattening, amplitude and the period of a free libration of the Moon, according to authors, are very uncertain. The authors didn't manage to identify completely the free libration of the Moon caused by the influence of a liquid core, the period, phase, and also structure of the arguments of free librations. Therefore here we continue research of effects of a liquid core of the Moon in its free librations.

Determination of the fourth mode of free libration of the Moon due to its liquid core. In this work, the approximate solution of a problem of free oscillations of a pole of a vector of angular velocity of rotation of the Moon and a pole of angular velocity of rotation of a liquid core (Poincaré's coordinate system given by a simple motion of the liquid) in relation to the principal axes of inertia was actually obtained. The solution was obtained in variations of Andoyer variables L, L_c and l, l_c (Ferrandiz and Barkin, 2000; Barkin et al., 2012). Here, L, L_c are projections of a vector of the angular momentum of the Moon $\mathbf{G}$ and the vector of the angular momentum of the relative motion of particles of the liquid $\mathbf{G}_c$ on a polar axis of inertia of the Moon $C\zeta$: $L = G\cos\theta$, $L_c = G_c\cos\theta_c$. Correspondingly, G and G_c are moduli of the specified vectors $\mathbf{G}$ and $\mathbf{G}_c$, θ and θ_c are angles, formed by vectors $\mathbf{G}$ and $\mathbf{G}_c$ with the polar axis of inertia of the Moon $C\zeta$.

The fourth mode of the free librations of the Moon is due to the hydrodynamic effect of the liquid core, and these canonical variables Andoyer determined by simple formulas. They follow from (17):

$$\boldsymbol{\delta}\mathbf{z} = S\left(\mathbf{z}^{(s)}\cos U_s - \mathbf{z}^{(s)*}\sin U_s\right), \tag{40}$$

where the frequency s expressed in terms of a combination of all the moments of inertia of the Moon and its core (24) or by the simplified formula (25) through the dynamic compressions of the liquid core ε_D, μ_D (Barkin et al., 2012):

$$U_s = s_0 t + U_s^{(0)}, \quad s_0 = n_F + n_F(\varepsilon_D + \mu_D)/2,$$
$$T_s = \frac{2\pi}{s_0} = T_F - \frac{2T_F}{\varepsilon_D + \mu_D}. \tag{41}$$

Frequency n_F, which is equal to the frequency of the corresponding argument of the theory of orbital motion of the Moon F, is equal to the unperturbed value of the angular velocity of rotation of the Moon, (one of the provisions of Cassini's laws). To this frequency there corresponds the draconic period of lunar orbital motion $T_F = 2\pi/n_F = 27.21222$ days. The free oscillation of the poles of the Moon with the quasi-diurnal (lunar) period $T_s = 27.19759$ days corresponds to the frequency of a free libration s_0. This period is not much shorter than the draconic period stated above.

For frequency s_0, we will use also the representation

$$s_0 = n_F + n_\Pi, \quad n_\Pi = n_F(\varepsilon_D + \mu_D)/2,$$
$$T_\Pi = \frac{2T_F}{\varepsilon_D + \mu_D}. \tag{42}$$

where frequency n_Π is a very low frequency of Poincaré, defining a long-periodic libration of the polar axis of rotation of the Moon due to the influence of the liquid core with a period T_Π.

Now we write the solution (40), (41) for the variations of the variables: $\delta L(\delta\theta)$, $\delta L_c(\delta\theta_c)$, δl, δl_c, introduced above, the variables defining oscillations of a pole of the Moon and its liquid core. This solution was constructed according to the method of the small parameter (Barkin, 1987) and describes the main dynamic effects in the rotation of the Moon caused by a liquid core (Barkin et al., 2012):

$$\delta L/Bn_F = -SL_0^{(s)*}\sin U_s, \quad \delta l = Sl_0^{(s)}\cos U_s, \tag{43}$$

$$\delta L_c/Bn_F = -SL_{c;0}^{(s)*}\sin U_s, \quad \delta l_c = Sl_{c;0}^{(s)}\cos U_s. \tag{44}$$

As true differential relations: $\delta L = -G\delta\theta$, $\delta L_c = -G_c\delta\theta_c$, the formulas for solving variational equations (43), (44) can be extended:

$$\delta\theta = SL_0^{(s)*}\sin U_s, \quad \delta l = Sl_0^{(s)}\cos U_s, \tag{45}$$

$$\delta\theta_c = \frac{E_c}{B}SL_{c;0}^{(s)*}\sin U_s, \quad \delta l_c = Sl_{c;0}^{(s)}\cos U_s. \tag{46}$$

Coefficients $L_0^{(s)*}$, $l_0^{(s)}$, $L_{c;0}^{(s)*}$ and $l_{c;0}^{(s)}$ in (45), (46) are expressed in terms of values of the second partial derivatives of a Hamiltonian of the problem of rotation of a two-layer Moon, the variables calculated at unperturbed values corresponding to motion according to Cassini's laws,

$$L_0^* = \frac{1}{s_0}\frac{\Delta_L}{\Delta}, \quad L_{c;0}^* = -\frac{1}{s_c}\frac{\Delta_{L_c}}{\Delta}, \quad l_{c;0} = -\frac{\Delta_{l_c}}{\Delta}\ (l_0 = 1), \tag{47}$$

where auxiliary notation is used

$$\Delta = n_F^2\left(\frac{B_c^2}{A_c C_c} - \frac{B_c B}{AC}\right) - s_0^2, \quad \Delta_L = -Bn_F^2\frac{BA_c}{A^2}s_0^2,$$
$$\Delta_{L_c} = -Bn_F^2\frac{B_c}{A}s_0^2, \quad \Delta_{l_c} = \frac{B}{A}n_F^2\left(\frac{A_c B}{AC_c} - \frac{B_c}{C_c}\right). \tag{48}$$

For the model of a dynamic structure of the Moon accepted in our work and its core it has the following values of the moments of inertia (Matsumoto et al., 2010; Barkin et al., 2012):

$$B = 0.393480000mr_0^2, B_c = E_c = 0.0001841759mr_0^2,$$
$$C = 0.393321295mr_0^2, C_c = D_c = 0.0001841094mr_0^2, \tag{49}$$
$$A = 0.393231806mr_0^2, A_c = F_c = 0.0001840444mr_0^2.$$

In (49) m and r_0 are the mass and the mean radius of the Moon. From the values of these parameters of the problem (49), we find the numerical values of the coefficients (47), (48):

$$L_0^* = -1.000011885, \quad L_{c;0}^* = 1.000095416, \quad l_{c;0} = -2133.74246 \ (l_0 = 1). \tag{50}$$

Correspondingly, the solution (45) and (46) can be written as:

$$\delta\theta = -1.000011885 S \sin U_s, \quad \delta l = S\cos U_s, \quad \delta\theta_c = 0.000468303 S\sin U_s, \quad \delta l_c = -2133.74246 S\cos U_s. \tag{51}$$

We emphasize that the unperturbed values of the Andoyer's variables correspond to the rotation of the Moon by Cassini's laws. In our notation, the axes of inertia of the Moon (in this paper) the moments of inertia of the Moon and its core satisfy the inequalities $B > C > A$ and $B_c > C_c > A_c$. Here, we introduce the new angular variable Π by the formula $U_s = F + \Pi$. For the initial values of the variables F_0, Π_0 and their frequencies, we have expressions:

$$U_s^{(0)} = F_0 + \Pi_0, \quad s_0 = n_F + n_\Pi, \quad U_s = s_0 t + F_0 + \Pi_0. \tag{52}$$

The liquid core causes free librations which are shown in all classical variables. To reveal them it is necessary to use equalities (35) and formulas of our solution in Andoyer's variables (33). For the considered variations they can be written in the form:

$$\begin{gathered}\delta\theta = SL_0^{(s)*}\sin U_s, \ \delta l = Sl_0^{(s)}\cos U_s,\\ \delta P_1^{\rm K} = -\cos\rho\,\delta l, \quad \delta P_2^{\rm K} = -\cos\rho\,\delta\theta,\\ \delta\rho_{\rm K} = \sin F\delta l + \cos F\delta\theta,\\ I\delta\sigma_{\rm K} = -\cos F\delta l + \sin F\delta\theta,\\ \delta\tau_{\rm K} = \tan\frac{\rho}{2}(\delta\theta\sin F - \delta l\cos F) = \tan\frac{\rho}{2}(I\delta\sigma_{\rm K}).\end{gathered} \tag{53}$$

In these equalities it is necessary to substitute free variations of Andoyer's variables, caused by a liquid core (45). As a result, for variations of classical variables of a problem (53) we obtain the following free librations due to an ellipsoidal liquid core:

$$\begin{aligned}\delta P_1^{\rm K} &= -\cos\rho\, S l_0^{(s)}\cos U_s,\\ \delta P_2^{\rm K} &= -\cos\rho\, S L_0^{(s)*}\sin U_s,\\ \delta\rho_{\rm K} &= \frac{1}{2}S\Big[\Big(L_0^{(s)*} + l_0^{(s)}\Big)\sin(U_s + F) + \Big(L_0^{(s)*} - l_0^{(s)}\Big)\sin(U_s - F)\Big],\\ I\delta\sigma_{\rm K} &= \frac{1}{2}S\Big[-\Big(L_0^{(s)*} + l_0^{(s)}\Big)\cos(U_s + F) + \Big(L_0^{(s)*} - l_0^{(s)}\Big)\cos(U_s - F)\Big],\\ \delta\tau_{\rm K} &= \frac{1}{2}S\tan\frac{\rho}{2}\Big[-\Big(L_0^{(s)*} + l_0^{(s)}\Big)\cos(U_s + F) + \Big(L_0^{(s)*} - l_0^{(s)}\Big)\cos(U_s - F)\Big].\end{aligned} \tag{54}$$

Substituting numerical values of parameters $L_0^* = -1.000011885$, $l_0 = 1$ variations (54) can be written in the following form:

$$\begin{aligned}\delta P_1^{\rm K} &= 0.999636588 S\cos U_s,\\ \delta P_2^{\rm K} &= 0.999648469 S\sin U_s\\ \delta\tau_{\rm K} &= -0.013481139 S\cos(U_s - F) + 0.798712306\times 10^{-7} S\cos(U_s + F),\\ \delta\rho_{\rm K} &= 1.0000059427 S\sin(U_s - F) + 0.0000059427 S\sin(U_s + F),\\ I\delta\sigma_{\rm K} &= -1.0000059427 S\cos(U_s - F) + 0.0000059427 S\cos(U_s + F).\end{aligned} \tag{55}$$

The solution of the problem of free librations of the Moon, caused by hydrodynamic influence of a liquid core, is obtained here in the system of coordinates connected with the Moon and with an ellipsoidal cavity of a core (in the system of the principal central axes of inertia of the Moon). For comparison of this analytical solution with the empirical theory (Rambaux and Williams, 2011) we will write down it in the system of coordinates $O\xi\eta\zeta$. For this, it follows that the fundamental argument U_s in formulas (53)–(55) be replaced with argument $U_s + \Omega$, considering a precession of the plane of a lunar orbit with an average angular velocity $n_\Omega < 0$ and with the corresponding period T_Ω.

In formulas (55), neglecting terms of order 10^{-4}–10^{-5} and setting $U_s = F + \Pi$, we will write down the solution (53) in the following form:

$$\begin{gathered}\delta P_1^{\rm K} = S\cos(F + \Omega + \Pi), \quad \delta P_2^{\rm K} = S\sin(F + \Omega + \Pi),\\ \delta\rho_{\rm K} = S\sin(\Omega + \Pi), \quad I\delta\sigma_{\rm K} = -S\cos(\Omega + \Pi),\\ \delta\tau = -0.01348 S\cos(\Omega + \Pi).\end{gathered} \tag{56}$$

The following task is the search for the possible free librations caused by a liquid ellipsoidal core,in the tables of the empirical theory of the Moon's physical librations (Rambaux and Williams, 2011), and their identification with the constructed perturbations (31)–(33).

Period, amplitude and phase of the fourth mode of the Moon's free librations caused by a liquid core. Tables of the empirical theory for classical variables of the theory of the physical librations of the Moon contain a large number of terms with certain amplitudes and periods, but whose nature is not certain (remains unknown). For 5 classical variables in the work of Rambaux and Williams (2011) there are about 50 similar terms. Table 3 of this article lists those that are characterized by the highest values of the amplitude. Designation of the status of variations Un in Tables 3 and 4 means that arguments for the corresponding librations of classical variables are unknown. Hydrodynamic influence of a liquid core on the physical librations of the Moon is one of major factors for the interpretation of unexplained terms of the empirical

Table 5. Determination of an initial phase Π_0 and amplitude S of the fourth mode of the free librations of the Moon, caused by influence of a liquid ellipsoidal core

$S\cos(\Omega_0+\Pi_0) = -0''052$	$S\sin(\Omega_0+\Pi_0) = 0''013$	$\Pi_0 = -139°$	$S = 0''054$
$S\cos(\Omega_0+\Pi_0) = 0''045$	$S\sin(\Omega_0+\Pi_0) = -0''002$	$\Pi_0 = -128°$	$S = 0''047$
$S\cos(F_0+\Omega_0+\Pi_0) = 0''000$	$S\sin(F_0+\Omega_0+\Pi_0) = 0''016$	$\Pi_0 = -132°$	$S = 0''016$
$S\cos(F_0+\Omega_0+\Pi_0) = -0''001$	$S\sin(F_0+\Omega_0+\Pi_0) = 0''016$	$\Pi_0 = -128°$	$S = 0''016$

theory (with the status Un). Therefore, they have to be analyzed first of all from the point of view of the two-layer model for free librations with the largest amplitudes from Tables 3 and 4.

Tables 3 and 4 contain designations of classical variables in the first column for the analysis of the free librations of the Moon. All librations here have an unknown origin (status). However, we were able to determine periods and amplitudes of these librations on the basis of long-series laser observations (Rambaux and Williams, 2011). Values of the periods of librations are given in days and amplitudes of librations in seconds of arc. Our purpose is to determine the period and to determine amplitude and an initial phase of the fourth mode of a free libration of the Moon (S and $U_s^{(0)} = F_0 + \Pi_0$ or Π_0). First, we note that the period of free nutation in the motion of a pole of the Moon due to the influence of the liquid core in the empirical theory (Rambaux and Williams, 2011) is determined in the ecliptic coordinate system $CXYZ$, associated with the equinox of date. Let us compare our analytical expressions (56) and empirical values of the amplitudes and periods of the librations of the Moon, caused by the liquid core (Table 2).

As a result, for the period of Poincaré $T_\Pi = 2\pi/n_\Pi$ we obtain the values:

$T_\Pi = 74470.00$ days (by perturbations on the variable $\delta\rho_K$),

$T_\Pi = 75797.75$ days (by perturbations on the variable $I\delta\sigma_K$),

$T_\Pi = 75133.87$ days (their average value).

And now, from the formula for Poincaré's period of oscillations of a pole of the Moon (16), the value of the sum of dynamic meridian compressions of a liquid core is determined: $\varepsilon_D + \mu_D = 7.244 \times 10^{-4}$. In the first section, the close estimate of this parameter was discussed $\varepsilon_D + \mu_D = 7.00 \times 10^{-4}$.

According to our analytical theory of rotation of the Moon, its physical librations are interpreted in the Cassini system coordinates $Cxyz$, which rotates relative to the coordinate system of the ecliptic $CXYZ$ in accordance with the laws of Cassini. Axis CZ and Cz of these coordinate systems coincide, but axis CX directed toward the equinox date, and axis Cx directed along the midline of the nodes of the lunar orbit on the ecliptic plane. This means that the coordinate axis Cx (and Cy) rotates in the ecliptic plane with an angular velocity $n_\Omega < 0$ in the main coordinate system, i.e., clockwise when viewed from the end of the axis (owing to regressive shift of middle node of the lunar orbit to the ecliptic plane). The angular velocity of this motion is equal to $n_\Omega = -69629''1123$, the period of this motion is 6798.526 days (Rambaux and Williams, 2011). This means that in a coordinate system $CXYZ$ the period of free libration due to the liquid core will correspond to the frequency $s_0 - n_\Omega$, i.e., it is equal to $T_s = 2\pi/(s_0 - n_\Omega)$. The corresponding argument of a free libration (the fourth mode) can be presented as $U_s = F + \Pi$, where $\Pi = n_\Pi t + \Pi_0$ ($\Pi_0 = -F_0 + U_s^{(0)}$ is the initial phase of the input variable). If the frequency s_0 defines a quasi-diurnal (lunar oscillations with the period close to the value of the draconic period in 27.21222 days), then frequency $n_\Pi = s_0 - n_F$ defines a long-period libration with a period of about 206 years. Based on the analysis of the unidentified librations of the Moon with the periods from Tables 3 and 4, we arrive at the conclusion that the observed perturbations of variables P_1 and P_2 correspond to a frequency $s_0 - n_\Omega$ and corresponding period $T_s = 2\pi/(s_0 - n_\Omega)$.

We will use designations of additional arguments (39) and variations of two classical variables $\delta\rho_K$ and $\sin\rho\delta\sigma_K$ from the solution (56) in the following form:

$$\begin{aligned}\delta\rho_K &= -S\cos(\Omega_0+\Pi_0)\sin\Xi \\ &\quad - S\sin(\Omega_0+\Pi_0)\cos\Xi, \\ \sin\rho\delta\sigma_K &= S\sin(\Omega_0+\Pi_0)\sin\Xi \\ &\quad - S\cos(\Omega_0+\Pi_0)\cos\Xi.\end{aligned} \quad (57)$$

In (57) the representation used was $\Omega + \Pi = \Xi + \Omega_0 + \Pi_0$, $\Xi = (n_\Omega + n_\Pi)t$. Ω_0 and Π_0 are initial values of the corresponding arguments for an initial epoch 2000.0. On the other hand, for considered variations according to laser observations we have a similar representation (Tables 3, 4):

$$\begin{aligned} \delta\rho_K &= 0''052\sin\omega_\rho t - 0''013\cos\omega_\rho t, \\ \sin\rho\delta\sigma_K &= -0''002\sin\omega_\sigma t - 0''045\cos\omega_\sigma t, \\ \omega_\rho &\approx \omega_\sigma. \end{aligned} \quad (58)$$

For frequencies ω_ρ and ω_σ the respective periods (in accordance with Tables 3, 4) have relatively similar values equal to 7481.5 and 7468.4 days. The average of these periods 7475.0 days we identify with a period of free libration $T_{\Omega+\Pi} = 2\pi/(n_\Omega + n_\Pi)$, due to the influence of the ellipsoidal liquid core. Assuming that the small differences in the values of these periods in the empirical theory (7481.5 and 7468.4 days) are caused by errors in the calculations, we take the observed value of the period,

$$\begin{aligned} T_{\Omega+\Pi} &= 2\pi/(n_\Omega + n_\Pi) \approx 2\pi/\omega_\rho \\ &\approx 2\pi/\omega_\sigma \approx 7475.0 \text{ days}. \end{aligned} \quad (59)$$

As a result of comparison of perturbations (57) and (58) we will obtain a system from four equations (see Table 5). From the theory of orbital motion of the Moon we have the initial value of the argument $\Omega_0 = 125°045$ (for the initial epoch 2000.0) (Simon et al., 1994; Kudryavtsev, 2007).

We will consider now variations of two other classical variables of the theory of physical librations of the Moon P_1^K and P_2^K from the solution (56), which are represented in the form:

$$\begin{aligned} \delta P_1^K &= S\cos(F_0 + \Omega_0 + \Pi_0)\cos\Sigma \\ &- S\sin(F_0 + \Omega_0 + \Pi_0)\sin\Sigma, \\ \delta P_2^K &= -S\sin(F_0 + \Omega_0 + \Pi_0)\cos\Sigma \\ &- S\cos(F_0 + \Omega_0 + \Pi_0)\sin\Sigma, \end{aligned} \quad (60)$$

where $\Sigma = (n_F + n_\Omega + n_\Pi)t$. On the other hand, for considered variations according to the empirical theory based on laser observations of the distances to the light reflectors on the Moon we have a similar representation (Rambaux and Williams, 2011) (Table 3):

$$\begin{aligned} \delta P_1^K &= 0''000\cos\omega_{P_1} t - 0''016\sin\omega_{P_1} t, \\ \delta P_2^K &= -0''016\cos\omega_{P_2} t + 0''001\sin\omega_{P_2} t. \end{aligned} \quad (61)$$

For variations of the long-period variables in 74470.00 and 75797.75 days (from Table 4) we obtain an average value of the period of 75133.87 days. For frequencies ω_{P_1} and ω_{P_2} in (61) corresponding to the periods (according to Table 4) have the same value equal to 27.312 days. Assuming that the small differences in the values of the frequencies ω_{P_1} and ω_{P_2} (and in the corresponding periods) are caused by errors in calculations, at once we obtain an assessment of the quasi-diurnal Poincaré period for a free libration of the Moon:

$$T_{F+\Omega+\Pi} = \frac{2\pi}{n_F + n_\Omega + n_\Pi} \approx \frac{2\pi}{\omega_{P_1}} \approx \frac{2\pi}{\omega_{P_2}} \approx 27.312 \text{ days} \quad (62)$$

Accepting values $\omega_{P_1} = \omega_{P_2} = n_F + n_\Omega + n_\Pi$ according to the Table 3, we will obtain in addition to four equations considered above, four more equations for determination of the amplitude and a phase of the fourth mode of free librations of the Moon (Table 5).

Nevertheless, the amplitude of this libration is smaller in magnitude in comparison with the theoretical value (see Table 4). This means that the specified characteristics of a libration of the Moon can depend on other possible factors and have to be studied in more detail and fully in the future, including new methods and approaches. For example, using the astrometry method of observations from the Moon's surface (Hanada et al., 2004). Theoretical studies of the forced and free librations of the Moon in the future have to be performed on the basis of better models of an internal structure of the Moon with a liquid and solid core (Barkin, 2011).

For determination of an initial phase Π_0 (for the epoch 2000.0 JD) and amplitude S of a free libration of the Moon, we have a system of eight equations with two unknowns (see the first two columns of Table 5), in which we must take the following initial values for the well-known arguments of the theory of orbital motion of the Moon: $F_0 = 93°27$, $\Omega_0 = 125°045$ (Kudryavtsev, 2007). In Table 5, for every line of values of amplitudes of variations, the corresponding values of an initial phase Π_0 and amplitude S are determined (they are presented in the third and fourth columns of the table). Values of a phase Π_0 are in degrees, and the amplitude of the fourth mode S in arcseconds.

It is clearly seen that the initial value of the phase of the fourth mode of the free librations of the Moon is determined on the basis of fairly confident laser observations (values in area $\Pi_0 = -128°...-139°$). The amplitude of this free librations of the Moon is less confident (with some scatter in the values $S = 0''02$–$0''05$). Thus, all 8 equations from Table 5, are approximately satisfied for an initial phase of a free nutation of a liquid core $\Pi_0 = -128°...-139°$ and for values of amplitude $S = 0''02$–$0''05$. The optimum values of these parameters determined by a method of least squares are $S = 0''0395$ and $\Pi_0 = -134°$.

CONCLUSION

The analytical theory of the forced and free librations of a two-layer model of the Moon (with a liquid ellipsoidal core) is developed. The main resonant librations of the Moon are determined in Andoyer's variables and in classical variables of the theory of physical librations in good agreement with the modern empirical theory constructed using the measurements of laser ranging over the last 40 years (Rambaux and Williams, 2011). At first, we determined the amplitude and the initial phase and the period of the fourth mode of free librations (Table 1) due to the liquid core. At the same time, an explanation and interpretation of previously identified terms of the empirical theory was given.

The period of a free libration of the Moon caused by a liquid ellipsoidal core was estimated by us at 75133.87 days or, respectively, 205.7 years. This period according to a formula for the period (29) corresponds to the sum of dynamic meridian compressions of the Moon in $\varepsilon_D + \mu_D = 7.244 \times 10^{-4}$. With the assumption of similarity of dynamic compressions of all the Moon and its core, the following estimates were obtained: $\varepsilon_D = 4.419 \times 10^{-4}$, $\mu_D = 2.825 \times 10^{-4}$. Eight free librations in classical variables of the theory of rotation of the Moon ρ, $I\sigma$ and P_1, P_2 from the general list of unidentified librations of the Moon identified as a result of the analysis of laser observation (Rambaux and Williams, 2011), obtained, in our work, an explanation and mechanical interpretation (Table 4). In addition, in the work the small free libration in longitude τ was found, also caused by the influence of a liquid ellipsoidal core, with a period of 7449.89 days and with a small amplitude of about 0″001 (Table 4).

In the future we plan to execute more wide-ranging studies of dynamic effects in rotation of the Moon due to its liquid core and solid core, including research using the dynamics of interacting gravitational shells of celestial bodies (Barkin, 2002; 2011; Barkin et al., 2012). The relevance of these researches is connected with the increasing accuracy of laser observation of the Moon and with the development and implementation of projects studying the rotation of the Moon directly from the surface of the Moon, in particular, in the Japanese ILOM project (Hanada et al., 2004). Results of this work also testify that the specified characteristics of libration of the Moon depend on other possible factors and have to be studied in more detail and more precisely in the future, including with use of new methods and approaches. For example, an astrometry method of observations from the Moon surface in the Japanese ILOM project (Hanada et al., 2004). Theoretical researches of the forced and free librations of the Moon in the future will be executed on the basis of more perfect models of an internal structure of the Moon with a liquid core and solid core (taking into account their eccentric position, gravitational interaction, oscillations of their centers of masses, etc. factors (Barkin, 2011)).

Results obtained in this paper provide new evidence and confirm the existence of a liquid core by an independent method based on dynamic studies of the rotation of the Moon.

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